DE RHAM COHOMOLOGY AND HODGE DECOMPOSITION FOR QUANTUM GROUPS

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Abstract

Let $\Gamma=\Gamma_{\tau,z}$ be one of the N^2-dimensional bicovariant first order differential calculi for the quantum groups $\mathrm{GL}_q(N)$, $\mathrm{SL}_q(N)$, $\mathrm{SO}_q(N)$, or $\mathrm{Sp}_q(N)$, where q is a transcendental complex number and z is a regular parameter. It is shown that the de Rham cohomology of Woronowicz' external algebra $\Gamma^\wedge$ coincides with the de Rham cohomologies of its left-coinvariant, its right-coinvariant and its (twosided) coinvariant subcomplexes. In the cases $\mathrm{GL}_q(N)$ and $\mathrm{SL}_q(N)$ the cohomology ring is isomorphic to the coinvariant external algebra $\Gamma^\wedge_{\mathrm{Inv}}$ and to the vector space of harmonic forms. We prove a Hodge decomposition theorem in these cases. The main technical tool is the spectral decomposition of the quantum Laplace-Beltrami operator.

1. *Introduction*

About ten years ago a general framework for covariant differential calculi on Hopf algebras was invented by Woronowicz [**18**]. Since then covariant first order differential calculi on quantum groups have been constructed, studied and classified by many authors, see for instance [**3, 1, 7, 13, 8**].

In classical differential geometry higher order differential forms naturally appear. The de Rham cohomology of a compact Lie group G characterises certain topological properties of G. However, there

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are only very few papers dealing with higher order differential calculi and de Rham cohomology on quantum groups. Maltsiniotis [**11**] constructed a multiparameter differential graded bialgebra of $\mathrm{GL}(N)$-type having the classical dimensions of the bigraded components. Tsygan [**16**] studied the linear $\mathrm{GL}_q(N)$-differential calculus in detail. The de Rham cohomologies of the left-covariant $3D$-calculus and of the bicovariant $4D_\pm$-calculus were determined by Woronowicz [**17**] and Grießl [**9**], respectively.

The present paper deals with the de Rham cohomology and the Hodge decomposition of the standard bicovariant differential calculi on the quantum groups of types A, B, C, and D. We use Woronowicz' construction of the external algebra. Our first main result (Theorem 3.1) says that the embeddings of the left-coinvariant, the right-coinvariant, and the coinvariant (both left- and right-coinvariant) subcomplex into the whole complex of differential forms are quasi-isomorphisms, respectively. This means that their de Rham cohomologies coincide. Our second main result (Theorem 3.2) is a Hodge decomposition theorem obtained for the quantum groups of type A. The main technical tool is the quantum Laplace-Beltrami operator [**6**] which is constructed using the dual pairing of two bicovariant differential calculi. Differential forms vanishing under the action of the quantum Laplace-Beltrami operator are called harmonic forms. If the parameter value z of the differential calculus is regular, then the following three spaces coincide: the de Rham cohomology ring, the algebra of coinvariant forms, and the vector space of harmonic forms. For a special class of non-regular parameter values z however, there exist additional harmonic forms like $\mathcal{D}^k\rho$, $k \in \mathbb{Z}$, where $\mathcal{D}$ is the quantum determinant and ρ is a coinvariant differential form.

In case of the orthogonal and symplectic quantum groups there exist harmonic forms which are not closed. Therefore we only have a

restricted Hodge decomposition for elements in the image of the quantum Laplace-Beltrami operator.

Our standing assumption is that the deformation parameter q is a transcendental complex number. On the one hand this ensures that the coordinate Hopf algebra $\mathcal{A}$ of the quantum group is cosemisimple and that the theory of corepresentations of $\mathcal{A}$ corresponds to the representation theory of the underlying classical Lie group. On the other hand it guarantees that there are no other harmonic functions except from polynomials in the quantum determinant.

This paper is organised as follows. In Section 2 we collect some basic definitions and preliminary facts needed later. The main result about quasi-isomorphisms is Theorem 3.1. The Hodge decomposition for $\mathrm{SL}_q(N)$ and $\mathrm{GL}_q(N)$ is given in Theorem 3.2. In Section 4 we prove the isomorphy of the left-dual and the right-dual Hopf bimodules and we add some properties of the contraction operator. Section 6 is devoted to the spectral decomposition of the quantum Laplace-Beltrami operator. Theorem 3.1 is proven therein. Section 7 deals with the duality of differential and codifferential operators. We use the notion of homomorphic differential calculi due to Pflaum and Schauenburg [**12**] and show that $\mathit{\Gamma}_{+,z}$ and $\mathit{\Gamma}_{-,z}$ are weakly isomorphic. The proof of Theorem 3.2 is given in Section 7.

We close this introduction by fixing some assumptions and notations that are used in the sequel. All vector spaces, algebras, bialgebras, etc. are meant to be $\mathbb{C}$-vector spaces, unital $\mathbb{C}$-algebras, $\mathbb{C}$-bialgebras etc. The linear span of a set $\{a_i | i \in K\}$ is denoted by $\langle a_i | i \in K\rangle$. The symbol $\mathcal{A}$ always denotes a Hopf algebra. We write $\mathcal{A}^*$ for the dual vector space of $\mathcal{A}$ and $\mathcal{A}^\circ$ for the dual Hopf algebra. All modules, comodules, and bimodules are assumed to be $\mathcal{A}$-modules, $\mathcal{A}$-comodules, and $\mathcal{A}$-bimodules if not specified otherwise. The comultiplication, the counit, and the antipode of $\mathcal{A}$ are denoted by Δ, ε, and by S, respectively. For a cosemisimple Hopf algebra $\mathcal{A}$, let h denote the Haar functional

on $\mathcal{A}$. Let $\boldsymbol{v} = (v^i_j)_{i,j\in K}$ be a corepresentation of $\mathcal{A}$. The coalgebra of matrix elements is denoted by $\mathcal{C}(\boldsymbol{v}) = \langle v^i_j \,|\, i,j \in K\rangle$. As usual $\boldsymbol{v}^{\mathrm{c}}$ denotes the contragredient corepresentation of $\boldsymbol{v}$, where $(v^{\mathrm{c}})^i_j = S(v^j_i)$. For the space of intertwiners of two corepresentations $\boldsymbol{v}$ and $\boldsymbol{w}$ the symbol $\mathrm{Mor}(\boldsymbol{v},\boldsymbol{w})$ is used. We write $\mathrm{Mor}(\boldsymbol{v})$ for $\mathrm{Mor}(\boldsymbol{v},\boldsymbol{v})$. We use the same notation $\mathrm{Mor}(\boldsymbol{f},\boldsymbol{g})$ for representations $\boldsymbol{f}$ and $\boldsymbol{g}$ of $\mathcal{A}$. If A is a linear mapping, A^t denotes the transpose (dual) mapping of A and $\mathrm{tr}\,A$ the trace of A. Lower indices of A always refer to the components of a tensor product where A acts ('leg numbering'). The unit matrix is denoted by I. Unless it is explicitly stated otherwise, we use Einstein's convention to sum over repeated indices in different factors. Throughout we assume that N is a positive integer with $N \geq 2$ for quantum groups of type A and $N \geq 3$ for quantum groups of types B, C, and D. The q-numbers we are dealing with are $[\![N]\!]_q := (q^N - q^{-N})/(q - q^{-1})$. We use Sweedler's notation $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$, $\Delta_{\mathrm{L}}(\rho) = \sum \rho_{(-1)} \otimes \rho_{(0)}$ and $\Delta_{\mathrm{R}}(\rho) = \sum \rho_{(0)} \otimes \rho_{(1)}$ for the coproduct, for left coactions and for right coactions, respectively. If $\mathcal{B}$ is an $\mathcal{A}$-bimodule then the mapping $b \triangleleft a := Sa_{(1)}\, b\, a_{(2)}$, $a \in \mathcal{A}$, $b \in \mathcal{B}$, is called the *right adjoint action* of $\mathcal{A}$ on $\mathcal{B}$.

2. *Preliminaries*

In this section we recall some general notions and facts from the theory of bicovariant differential calculus [**18**], which are needed later. More details and proofs of related or unproven statements can be found in [**8**, Chapter 14].

Hopf bimodules and bicovariant first order differential calculi. A *Hopf bimodule (bicovariant bimodule)* over $\mathcal{A}$ is a bimodule $\mathit{\Gamma}$ together with linear mappings $\Delta_{\mathrm{L}}\colon \mathit{\Gamma} \to \mathcal{A} \otimes \mathit{\Gamma}$ and $\Delta_{\mathrm{R}}\colon \mathit{\Gamma} \to \mathit{\Gamma} \otimes \mathcal{A}$ such that $(\mathit{\Gamma}, \Delta_{\mathrm{L}}, \Delta_{\mathrm{R}})$ is a bicomodule, $\Delta_{\mathrm{L}}(a\omega b) = \Delta(a)\Delta_{\mathrm{L}}(\omega)\Delta(b)$, and $\Delta_{\mathrm{R}}(a\omega b) = \Delta(a)\Delta_{\mathrm{R}}(\omega)\Delta(b)$ for a, $b \in \mathcal{A}$ and $\omega \in \mathit{\Gamma}$. We call the elements of the vector spaces $\mathit{\Gamma}_{\mathrm{L}} := \{\omega | \Delta_{\mathrm{L}}(\omega) = 1 \otimes \omega\}$ and $\mathit{\Gamma}_{\mathrm{R}} :=$

$\{\omega|\Delta_{\mathrm{R}}(\omega)=\omega\otimes 1\}$ *left-coinvariant* and *right-coinvariant*, respectively. The elements of $\Gamma_{\mathrm{Inv}}=\Gamma_{\mathrm{L}}\cap\Gamma_{\mathrm{R}}$ are called *coinvariant*. The dimension of the Hopf bimodule Γ is defined to be the dimension of the vector space Γ_{L}. We always assume that Γ is finite dimensional.

For $\rho\in\Gamma$ and $f\in\mathcal{A}^*$ we define $f*\rho=\rho_{(0)}f(\rho_{(1)})$ and $\rho*f=f(\rho_{(-1)})\rho_{(0)}$. In this way left and right actions of $\mathcal{A}^*$ on Γ are defined. If $\{\omega_i\}$ is a basis of the vector space Γ_{L} then there exist matrices $\boldsymbol{v}=(v^i_j)$, $v^i_j\in\mathcal{A}$, and $\boldsymbol{f}=(f^i_j)$, $f^i_j\in\mathcal{A}^\circ$, such that $\boldsymbol{v}$ is a corepresentation of $\mathcal{A}$, $\boldsymbol{f}$ is a representation of $\mathcal{A}$, $\Delta_{\mathrm{R}}(\omega_j)=\omega_i\otimes v^i_j$, and $\omega_i\triangleleft a=f^i_j(a)\omega_j$. We briefly write $\Gamma=(\boldsymbol{v},\boldsymbol{f})$ in this situation.

A *first order differential calculus* over $\mathcal{A}$ (FODC for short) is an $\mathcal{A}$-bimodule Γ with a linear mapping $\mathrm{d}\colon\mathcal{A}\to\Gamma$ which satisfies the Leibniz rule $\mathrm{d}(ab)=\mathrm{d}a{\cdot}b+a{\cdot}\mathrm{d}b$ for $a,\,b\in\mathcal{A}$, such that Γ is the linear span of elements $a\,\mathrm{d}b$ with $a,b\in\mathcal{A}$. A FODC Γ is called *left-covariant* if there exists a linear mapping $\Delta_{\mathrm{L}}\colon\Gamma\to\mathcal{A}\otimes\Gamma$ such that $\Delta_{\mathrm{L}}(a\mathrm{d}b)=\Delta(a)(\mathrm{id}\otimes\mathrm{d})\Delta(b)$ for $a,b\in\mathcal{A}$. Similarly, Γ is called *right-covariant* if there exists a linear mapping $\Delta_{\mathrm{R}}\colon\Gamma\to\Gamma\otimes\mathcal{A}$ such that $\Delta_{\mathrm{R}}(a\mathrm{d}b)=\Delta(a)(\mathrm{d}\otimes\mathrm{id})\Delta(b)$ for $a,b\in\mathcal{A}$. The FODC Γ is called *bicovariant* if it is both left- and right-covariant. In this case $(\Gamma,\Delta_{\mathrm{L}},\Delta_{\mathrm{R}})$ is a Hopf bimodule.

Let Γ be a left-covariant FODC. A central role plays the mapping $\omega\colon\mathcal{A}\to\Gamma_{\mathrm{L}}$ defined by $\omega(a)=Sa_{(1)}\mathrm{d}a_{(2)}$. The vector space $\mathcal{R}=\ker\varepsilon\cap\ker\omega$ is a right ideal of $\ker\varepsilon$. It is called the *associated right ideal* to the left-covariant FODC Γ. Suppose that $\{\omega_i\,|\,i\in K\}$ is a linear basis of Γ_{L}. Then there exist linear functionals $X_i\in\mathcal{A}^\circ$, $i\in K$, such that $\omega(a)=\sum_{i\in K}X_i(a)\omega_i$ for $a\in\mathcal{A}$. The linear space $\mathcal{X}=\langle X_i\,|\,i\in K\rangle$ is called the *quantum tangent space* of Γ. We have $\mathrm{d}a=\sum_{i\in K}(X_i*a)\omega_i$ for $a\in\mathcal{A}$.

Exterior Algebras. We briefly recall Woronowicz' construction of the external algebra to a given Hopf bimodule Γ. Obviously $\Gamma^\otimes=\bigoplus_{k\geq 0}\Gamma^{\otimes k}$ is again a Hopf bimodule. Let Λ be another Hopf bimodule.

Then there exists a unique homomorphism $\sigma\colon \Gamma\otimes_{\mathcal{A}}\Lambda \to \Lambda\otimes_{\mathcal{A}}\Gamma$ of Hopf bimodules called the *braiding* with

$$\sigma(\alpha\otimes_{\mathcal{A}}\beta) = \beta_{(0)}\otimes_{\mathcal{A}}(\alpha\triangleleft\beta_{(1)}), \quad \sigma^{-1}(\alpha\otimes_{\mathcal{A}}\beta) = \beta\triangleleft(S^{-1}\alpha_{(1)})\otimes_{\mathcal{A}}\alpha_{(0)} \tag{1}$$

for $\alpha \in \Gamma_{\mathrm{L}}$ and $\beta \in \Lambda_{\mathrm{L}}$, see [**8**, Subsection 13.1.4]. Moreover, σ satisfies the braid equation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$. Let S_k be the symmetric group of k elements and let s_n denote the simple transposition of n and $n+1$. For $\pi \in \mathrm{S}_k$ the expression $\pi = s_{i_1}\cdots s_{i_r}$ is called reduced if r is minimal. Since σ satisfies the braid equation, the bimodule automorphism $\sigma_\pi\colon \Gamma^{\otimes k} \to \Gamma^{\otimes k}$, $\sigma_\pi = \sigma_{i_1}\cdots\sigma_{i_r}$, does not depend on the choice of the reduced expression for π. Define the antisymmetriser A_k, $k \geq 1$, $A_0 = \mathrm{id}$, and the endomorphism $B_{i,j}$, $i+j=k$, of $\Gamma^{\otimes k}$ by

$$A_k = \sum_{\pi\in\mathrm{S}_k} \mathrm{sgn}(\pi)\sigma_\pi \quad \text{and} \quad B_{i,j} = \sum_{\pi^{-1}\in\mathrm{C}_{i,j}} \mathrm{sgn}(\pi)\sigma_\pi, \tag{2}$$

where $\mathrm{C}_{i,j} = \{\pi \in \mathrm{S}_{i+j} \,|\, \pi(1) < \cdots < \pi(i), \pi(i+1) < \cdots < \pi(i+j)\}$ are the shuffle permutations. We have $B_{1,i} = \mathrm{id} - \sigma_1 + \sigma_1\sigma_2 - \ldots + (-1)^i\sigma_1\cdots\sigma_i$ and $B_{i,1} = \mathrm{id} - \sigma_i + \sigma_i\sigma_{i-1} - \ldots + (-1)^i\sigma_i\cdots\sigma_1$. The above constructions are possible for any bimodule isomorphism σ which satisfies the braid relation. Therefore, replacing σ everywhere by σ^{-1} the above works as well. In what follows we will use both kinds of operators and write $A_k^{\pm}$ and $B_{i,j}^{\pm}$ whenever we are dealing with $\sigma^{\pm 1}$. Now we define *Woronowicz' external algebra* $\Gamma^\wedge = \bigoplus_{k\geq 0}\Gamma^{\wedge k}$ by $\Gamma^{\wedge k} := \Gamma^{\otimes k}/\ker A_k$. It can be shown that $\Gamma^\wedge$ is both a Hopf bimodule and a graded super Hopf algebra, cf. [**8**, Subsection 13.2.2].

σ-metrics and contractions. We recall from [**6**, Section 2] the important notion of a σ-metric. Let (Γ_+, Γ_-) be a pair of Hopf bimodules and $\overline{\Gamma} := \Gamma_+\otimes_{\mathcal{A}}\Gamma_- \oplus \Gamma_-\otimes_{\mathcal{A}}\Gamma_+$. A linear mapping $g\colon \overline{\Gamma} \to \mathcal{A}$ is called a *σ-metric* of (Γ_+, Γ_-) if g is a homomorphism of $\mathcal{A}$-bimodules, g is non-degenerate, g is σ-symmetric, i. e. $g{\circ}\sigma = g$, and

$$g_{12}\sigma_{23}\sigma_{12} = g_{23}, \quad g_{23}\sigma_{12}\sigma_{23} = g_{12} \tag{3}$$

in $\varLambda\otimes_{\mathcal{A}}\overline{\varGamma}$ and $\overline{\varGamma}\otimes_{\mathcal{A}}\varLambda$, respectively. Here $\varLambda$ denotes either $\varGamma_+$ or $\varGamma_-$. The σ-metric g is called *bicovariant* if

$$\Delta\circ g = (\mathrm{id}\otimes g)\Delta_{\mathrm{L}} = (g\otimes\mathrm{id})\Delta_{\mathrm{R}} \tag{4}$$

in $\varGamma$. In this paper we are only concerned with bicovariant σ-metrics. It is easily seen that (3) follows from the bicovariance [**2**, Section II. 4]. By abuse of notation we sometimes skip the tensor sign and write $g(\xi a,\zeta) = g(\xi, a\zeta) = g(\xi a\otimes_{\mathcal{A}}\zeta)$. We recursively extend g to an endomomorphism $\widetilde{g}$ of the Hopf bimodule $\varGamma_\tau^{\otimes}\otimes_{\mathcal{A}}\varGamma_{-\tau}^{\otimes}$, $\tau\in\{+,-\}$, by

$$\begin{gathered}\widetilde{g}(\xi,a)=\xi a,\quad \widetilde{g}(a,\zeta)=a\zeta,\\ \widetilde{g}(\xi\otimes_{\mathcal{A}}\rho,\eta\otimes_{\mathcal{A}}\zeta)=\widetilde{g}(\xi\, g(\rho,\eta),\zeta)\end{gathered} \tag{5}$$

for $\xi\in\varGamma_\tau^{\otimes}$, $\rho\in\varGamma_\tau$, $\zeta\in\varGamma_{-\tau}^{\otimes}$, $\eta\in\varGamma_{-\tau}$, and $a\in\mathcal{A}$. If ξ and ζ are of degree n and k, $n\geq k$, then $\widetilde{g}(\xi,\zeta)\in\varGamma_\tau^{\otimes n-k}$.

Next we define *contractions* $\langle\cdot,\cdot\rangle_\pm : \varGamma_\tau^{\otimes k}\otimes_{\mathcal{A}}\varGamma_{-\tau}^{\otimes l}\to\varGamma_{\tau'}^{\otimes|k-l|}$, $\tau\in\{+,-\}$, where $\tau'=\tau$ for $k\geq l$, and $\tau'=-\tau$ for $k<l$, by

$$\begin{aligned}\langle\xi,\zeta\rangle_\pm &:= \widetilde{g}(B^\pm_{k-l,l}\xi\,,\,A^\pm_l\zeta)\quad\text{for}\quad k\geq l,\\ \langle\xi,\zeta\rangle_\pm &:= \widetilde{g}(A^\pm_k\xi\,,\,B^\pm_{k,l-k}\zeta)\quad\text{for}\quad k<l.\end{aligned} \tag{6}$$

Since g is, the map $\langle\cdot,\cdot\rangle_\pm$ is a homomorphism of Hopf bimodules as well. If both k and l are less than two the contraction does not depend on the sign $\pm$ and we sometimes omit it, $\langle\xi,\zeta\rangle_+ = \langle\xi,\zeta\rangle_- =: \langle\xi,\zeta\rangle$.

The next property shows that the antisymmetriser is symmetric with respect to $\widetilde{g}$. For nonnegative integers i,j,k,l, with $1\leq i+j\leq k,l$, we have

$$\widetilde{g}\circ\big((\mathrm{id}^{\otimes k-i-j}\otimes A_i^\pm\otimes\mathrm{id}^{\otimes j}),\mathrm{id}^{\otimes l}\big) = \widetilde{g}\circ\big(\mathrm{id}^{\otimes k},(\mathrm{id}^{\otimes j}\otimes A_i^\pm\otimes\mathrm{id}^{\otimes l-i-j})\big).$$

Therefore the definition of $\langle\cdot,\cdot\rangle_\pm$ can be extended to a contraction map of *exterior algebras* namely to $\langle\cdot,\cdot\rangle_\pm : \varGamma_\tau^{\wedge k}\otimes_{\mathcal{A}}\varGamma_{-\tau}^{\wedge l}\to\varGamma_{\tau'}^{\wedge|k-l|}$, $\tau\in\{+,-\}$, where $\tau'=\tau$ for $k\geq l$ and $\tau'=-\tau$ for $k<l$.

Differential Calculi on Quantum Groups. Let $\mathcal{A}$ be the coordinate Hopf algebra $\mathcal{O}(G_q)$ of one of the quantum groups $\mathrm{GL}_q(N)$, $\mathrm{SL}_q(N)$,

$\mathrm{O}_q(N)$, $\mathrm{SO}_q(N)$, or $\mathrm{Sp}_q(N)$ as defined in [**4**, Section 1]. Let $\boldsymbol{u} = (u^i_j)_{i,j=1,\ldots,N}$ be the fundamental matrix corepresentation of $\mathcal{A}$. The corresponding $\hat{R}$-matrix for the A-series is given in [**4**, Subsection 1.2]. The $\hat{R}$-matrix for the B, C, and D series as well as the defining the antipode matrix C are given in [**4**, Subsection 1.4]. Now we define the invertible diagonal matrix $D = (d_i\delta_{ij})$, $D \in \mathrm{Mor}(\boldsymbol{u}^{\mathrm{cc}}, \boldsymbol{u})$. In case of $\mathrm{GL}_q(N)$ and $\mathrm{SL}_q(N)$ we set $d_i = q^{N+1-2i}$ and $\mathfrak{r} = q^N$. For the B, C, and D series set $D = C^tC^{-1}$ and $\mathfrak{r} = \epsilon q^{N-\epsilon}$, where $\epsilon = 1$ in the orthogonal case and $\epsilon = -1$ in the symplectic case. Let $\mathfrak{s} = \operatorname{tr} D = \operatorname{tr} D^{-1}$. For the quantum groups $\mathrm{GL}_q(N)$ and $\mathrm{O}_q(N)$ there exists a nontrivial group-like central element $\mathcal{D}$ of $\mathcal{A}$, the quantum determinant. It corresponds to the Young diagram (1^N) and can be constructed using the q-antisymmetric tensor [**5**, Section 5]. Note that $\mathcal{D}^2 = 1$ for $\mathrm{O}_q(N)$.

A complex number $x \in \mathbb{C}$ is called *admissible* for $\mathcal{A}$ if x is nonzero for $\mathrm{GL}_q(N)$, $x^N = q$ in case $\mathrm{SL}_q(N)$, $x^2 = 1$ in cases $\mathrm{O}_q(N)$, $\mathrm{SO}_q(2n)$, and $\mathrm{Sp}_q(N)$, and finally, $x = 1$ in case $\mathrm{SO}_q(2n+1)$. Recall that $\mathcal{A}$ is a coquasitriangular Hopf algebra (see [**8**, Subsection 10.1]) with universal r-form $\boldsymbol{r}_x$, given by $\boldsymbol{r}_x(u^i_j, u^k_l) = x^{-1}\hat{R}^{ki}_{jl}$, where x is an admissible parameter. The matrices $\boldsymbol{\ell}^+$ and $\boldsymbol{\ell}^-$ of representative functionals on $\mathcal{A}$ are defined by $\ell^{+i}_{\ j}(a) = \boldsymbol{r}_x(a, u^i_j)$ and $\ell^{-i}_{\ j}(a) = \boldsymbol{r}_y(S(u^i_j), a)$, x, y admissible for $\mathcal{A}$. A complex number z is called 2-*admissible* for $\mathcal{A}$ if $z = xy$ for two admissible numbers x and y for $\mathcal{A}$. Throughout the paper we assume z to be 2-admissible with fixed admissible numbers x, y and $z = xy$. Then the Hopf bimodule $\Gamma_{\tau,z}$, $\tau \in \{+,-\}$, is given as follows. Let $\{\omega^\tau_{ij} \,|\, i,j = 1,\ldots,N\}$ be a linear basis of the space of left-coinvariant forms of $\Gamma_{\tau,z}$. Define the right coaction $\Delta_{\mathrm{R}}(\rho)$ and the right adjoint action $\rho \triangleleft a$ for $\rho \in (\Gamma_{\tau,z})_{\mathrm{L}}$ and $a \in \mathcal{A}$ by

$$\Delta_{\mathrm{R}}(\omega^+_{ij}) = \omega^+_{kl} \otimes (\boldsymbol{u} \otimes \boldsymbol{u}^{\mathrm{c}})^{kl}_{ij}, \qquad \Delta_{\mathrm{R}}(\omega^-_{ij}) = \omega^-_{kl} \otimes (\boldsymbol{u}^{\mathrm{cc}} \otimes \boldsymbol{u}^{\mathrm{c}})^{kl}_{ij}, \tag{7}$$

$$\begin{aligned}
\omega^+_{ij} \triangleleft a &= S(\ell^{-k}_{\ i})\ell^{+j}_{\ l}(a)\omega^+_{kl} &&= \boldsymbol{r}_y(u^k_i, a_{(1)})\boldsymbol{r}_x(a_{(2)}, u^j_l)\omega^+_{kl},\\
\omega^-_{ij} \triangleleft a &= S^{-1}(\ell^{+k}_{\ i})\ell^{-j}_{\ l}(a)\omega^-_{kl} &&= \boldsymbol{r}_y(a_{(1)}, S(u^k_i))\boldsymbol{r}_x(S(u^j_l), a_{(2)})\omega^-_{kl}.
\end{aligned}$$

In shorthand notation we set $\Gamma_{+,z} = (\boldsymbol{u} \otimes \boldsymbol{u}^{\mathrm{c}}, \boldsymbol{\ell}^{-\mathrm{c}} \otimes \boldsymbol{\ell}^{+})$ and $\Gamma_{-,z} = (\boldsymbol{u}^{\mathrm{cc}} \otimes \boldsymbol{u}^{\mathrm{c}}, {}^{\mathrm{c}}\boldsymbol{\ell}^{+} \otimes \boldsymbol{\ell}^{-})$, where $({}^{\mathrm{c}}\boldsymbol{f})^i_j := S^{-1}(f^j_i)$. There are unique up to scalars coinvariant 1-forms $\omega^+_0 = \sum_{i=1}^N \omega^+_{ii} \in \Gamma_{+,z}$ and $\omega^-_0 = \sum_{i=1}^N d_i^{-1} \omega^-_{ii} \in \Gamma_{-,z}$. In particular we have the following right adjoint actions

$$\omega^+_{ij} \triangleleft u^m_n = z^{-1} \hat{R}^{mk}_{iv} \hat{R}^{jv}_{nl}\, \omega^+_{kl}, \qquad \omega^+_0 \triangleleft u^m_n = z^{-1} (\hat{R}^2)^{mk}_{nl}\, \omega^+_{kl},$$
$$\omega^-_{ij} \triangleleft u^m_n = z d_i d_k^{-1} (\hat{R}^{-1})^{mk}_{iv} (\hat{R}^{-1})^{jv}_{nl}\, \omega^-_{kl}, \quad \omega^-_0 \triangleleft u^m_n = z d_k^{-1} (\hat{R}^{-2})^{mk}_{nl}\, \omega^-_{kl}.$$

Recall that $\Gamma^{\wedge}_{\tau,z}$ is an *inner* bicovariant differential calculus with coinvariant 1-form ω^τ_0, that is, the differential d_τ is given by

$$\mathrm{d}_\tau \rho = \omega^\tau_0 \wedge \rho - (-1)^k \rho \wedge \omega^\tau_0, \quad \rho \in \Gamma^{\wedge k}_\tau. \tag{8}$$

Projecting this equation for $\rho = a \in \mathcal{A}$ to the left-coinvariant part of $\Gamma_{\tau,z}$ we get

$$\omega^\tau(a) = \omega^\tau_0 \triangleleft a - \varepsilon(a) \omega^\tau_0, \tag{9}$$

where ω^τ denotes the ω-mapping for $\Gamma_{\tau,z}$. Let $\{X^\tau_{ij} \,|\, i,j = 1,\dots,N\}$ be the basis of the quantum tangent space $\mathcal{X}^\tau$ of $\Gamma_{\tau,z}$ dual to $\{\omega^\tau_{ij}\}$. Explicitly, we have

$$X^+_{ij} = S(\ell^{-i}{}_k) \ell^{+k}{}_j - \delta_{ij}, \qquad X^+_0 = \mathfrak{s}^{-1} (D^{-1})^i_j X^+_{ij}, \tag{10}$$
$$X^-_{ij} = d_i^{-1} \big(S(\ell^{+i}{}_k) \ell^{-k}{}_j - \delta_{ij} \big), \qquad X^-_0 = \mathfrak{s}^{-1} \textstyle\sum_i X^-_{ii}. \tag{11}$$

Here X^τ_0 denotes the dual basis element to ω^τ_0 with respect to the decomposition of Δ_{R} on $\Gamma_{\tau,z}$ into irreducible corepresentations. The corresponding projections in $\mathrm{Mor}(\boldsymbol{u} \otimes \boldsymbol{u}^{\mathrm{c}})$ and $\mathrm{Mor}(\boldsymbol{u}^{\mathrm{cc}} \otimes \boldsymbol{u}^{\mathrm{c}})$ are

$$(P^+_0)^{ij}_{kl} = \frac{1}{\mathfrak{s}} d_k^{-1} \delta_{ij} \delta_{kl} \quad \text{and} \quad (P^-_0)^{ij}_{kl} = \frac{1}{\mathfrak{s}} d_i^{-1} \delta_{ij} \delta_{kl}, \tag{12}$$

respectively. Note that X^τ_0 is central in $\mathcal{A}^\circ$ and $S(X^+_0) = X^-_0$. We set $X_0 := X^+_0 + X^-_0$.

It was shown in [**6**, Propositions 3.1, 3.3, and 3.4] that the settings

$$g(a\omega^+_{ij}, \omega^-_{kl}) = a D^j_k (D^{-1})^l_i \quad \text{and} \quad g(a\omega^-_{ij}, \omega^+_{kl}) = a \delta_{jk} \delta_{il} \tag{13}$$

define a bicovariant σ-metric of $(\varGamma_{+,z}, \varGamma_{-,z})$. Note that

$$g(\omega_0^\tau, \omega_0^{-\tau}) = \mathfrak{s}. \tag{14}$$

To simplify notations we sometimes write $\varGamma_\tau$ instead of $\varGamma_{\tau,z}$.

Quantum Laplace-Beltrami Operators. Our main technical tool to reduce the de Rham cohomology from $\varGamma^\wedge$ to the essentially smaller complex $\varGamma^\wedge_{\mathrm{Inv}}$ is the quantum Laplace-Beltrami operator which is defined below. For a slightly different notion see also [**6**, Section 6].

The mappings $\partial^\pm_\tau : \varGamma_\tau^{\wedge k} \to \varGamma_\tau^{\wedge k-1}$, $k \geq 0$, defined by $\partial^\pm_\tau(a) = 0$ for $a \in \mathcal{A}$ and

$$\partial^\pm_\tau \rho = \langle \rho, \omega_0^{-\tau} \rangle_\pm + (-1)^k \langle \omega_0^{-\tau}, \rho \rangle_\pm$$

for $\rho \in \varGamma_\tau^{\wedge k}$, $k > 0$, are called *codifferential operators on* $\varGamma_\tau^{\wedge k}$. The linear mappings $L^\pm_\tau : \varGamma_\tau^{\wedge k} \to \varGamma_\tau^{\wedge k}$ defined by

$$L^\pm_\tau := -\mathrm{d}_\tau \, \partial^\pm_\tau + \partial^\pm_\tau \, \mathrm{d}_\tau \tag{15}$$

are called *quantum Laplace-Beltrami operators*. The elements of the vector space

$$H^\pm(\varGamma_\tau^{\wedge k}) := \{\rho \in \varGamma_\tau^{\wedge k} \mid L^\pm_\tau \rho = 0\} \tag{16}$$

are called *harmonic k-forms*. If no confusion can arise we sometimes write $L^\pm$ instead of $L^\pm_\tau$.

On elements $a \in \mathcal{A}$ the Laplace-Beltrami operators defined in [**6**] and the operators $L^\pm_\tau$ coincide, since $\partial^\pm_\tau(a) = 0$. Therefore we have

$$L^+_\tau a = L^-_\tau a = -2\mathfrak{s}a + \langle \omega_0^+ a, \omega_0^- \rangle + \langle \omega_0^- a, \omega_0^+ \rangle$$

for $a \in \mathcal{A}$.

3. *Main Results*

Define the sets $P_+ = \{\lambda = (\lambda_1, \dots, \lambda_N) \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N,\ \lambda_i \in \mathbb{Z}\}$ and $P_{++} = \{\lambda \in P_+ \mid \lambda_N \geq 0\}$. In case of $\mathrm{GL}_q(N)$ the set P_+

parametrises exactly the irreducible corepresentations of $\mathcal{A}$, see [**8**, Theorem 11.51], where $(1,0,\dots,0)$, $(0,\dots,0,-1)$, and $(1,1,\dots,1)$ correspond to $\boldsymbol{u}$, $\boldsymbol{u}^{\mathrm{c}}$, and the determinant $\mathcal{D}$, respectively. We extend the notation [**10**, Chapter 1] for partitions $\lambda\in P_{++}$ to Young frames with "negative" columns: for $\lambda\in P_+$ define $|\lambda|=\lambda_1+\dots+\lambda_N$. For $i\in\{1,\dots,N\}$ and $j\in\mathbb{Z}$ we write $(i,j)\in\lambda$ if $1\le j\le\lambda_i$ or $\lambda_i<j\le 0$. In this situation with $x:=(i,j)$ set $\mathrm{sgn}(x):=1$ if $j\ge 1$ and $\mathrm{sgn}(x):=-1$ otherwise. Define the content $c(x):=j-i$ and $c(\lambda):=\sum_{x\in\lambda}\mathrm{sgn}(x)\,c(x)$. In particular $\sum_{x\in\lambda}\mathrm{sgn}(x)=|\lambda|$.

In case of $\mathrm{GL}_q(N)$ a complex parameter z is said to be *regular* if for all λ, $\mu\in P_+$ the number

$$\begin{aligned}F_{\lambda\mu}&=(z^{-|\mu|}-2+z^{|\lambda|})[\![N]\!]_q+(q-q^{-1})\times\\&\quad\times\big(z^{-|\mu|}\sum_{x\in\mu}\mathrm{sgn}(x)q^{N+2c(x)}-z^{|\lambda|}\sum_{x\in\lambda}\mathrm{sgn}(x)q^{-N-2c(x)}\big)\end{aligned}\tag{17}$$

is nonzero, except for the case $\lambda=\mu=(0)$.

Let $\varLambda^\wedge=\bigoplus_{k\ge 0}\varLambda^k$ be a differential graded algebra. As usual

$$H_{\mathrm{de\ R}}(\varLambda^\wedge)=\bigoplus_{k\ge 0}H^k_{\mathrm{de\ R}}(\varLambda^\wedge),\qquad H^k_{\mathrm{de\ R}}(\varLambda^\wedge)=\ker\mathrm{d}_k/\operatorname{im}\mathrm{d}_{k-1},\tag{18}$$

denotes the de Rham cohomology of $\varLambda^\wedge$. By the Leibniz rule $\ker\mathrm{d}=\bigoplus_{k\ge 0}\ker\mathrm{d}_k$ is a subalgebra of $\varLambda^\wedge$ and $\operatorname{im}\mathrm{d}=\bigoplus_{k\ge 0}\operatorname{im}\mathrm{d}_k$ is an ideal in $\ker\mathrm{d}$. Hence $H_{\mathrm{de\ R}}(\varLambda^\wedge)$ is an algebra. Since the differential d is bicovariant, the algebras $\varGamma^\wedge_{\mathrm{L}}$, $\varGamma^\wedge_{\mathrm{R}}$, and $\varGamma^\wedge_{\mathrm{Inv}}$, and the vector spaces $\varGamma^\wedge(\mathbf{1},\mathcal{D}):=\{\rho\in\varGamma^\wedge_{\mathrm{L}}\mid\Delta_{\mathrm{R}}\rho=\rho\otimes\mathcal{D}\}$ and $\varGamma^\wedge(\mathcal{D},\mathbf{1}):=\{\rho\in\varGamma^\wedge_{\mathrm{R}}\mid\Delta_{\mathrm{L}}\rho=\mathcal{D}\otimes\rho\}$ are $\mathbb{Z}$-graded differential complexes.

THEOREM 3.1. *Suppose that q is a transcendental complex number.* (a) *Let G_q denote one of the quantum groups $\mathrm{GL}_q(N)$, $\mathrm{SL}_q(N)$, $\mathrm{SO}_q(N)$, or $\mathrm{Sp}_q(N)$, and $\mathcal{A}=\mathcal{O}(G_q)$ its coordinate Hopf algebra. Let $\varGamma$ be one of the N^2-dimensional bicovariant first order differential calculi $\varGamma_{\tau,z}$ over $\mathcal{A}$, where z is 2-admissible. In the case $\mathrm{GL}_q(N)$ we assume in addition*

that z is regular. Then we have canonical isomorphisms

$$H_{\mathrm{de\ R}}(\Gamma^{\wedge}) \cong H_{\mathrm{de\ R}}(\Gamma^{\wedge}_{\mathrm{L}}) \cong H_{\mathrm{de\ R}}(\Gamma^{\wedge}_{\mathrm{R}}) \cong H_{\mathrm{de\ R}}(\Gamma^{\wedge}_{\mathrm{Inv}}).$$

(b) *Let G_q be one of the quantum groups $\mathrm{GL}_q(N)$ or $\mathrm{O}_q(2n+1)$, $\mathcal{A} = \mathcal{O}(G_q)$, and Γ as above. In the case $\mathrm{GL}_q(N)$ we assume that $z^N q^{-2} = \zeta$, where ζ is a primitive m^{th} root of unity, $m \in \mathbb{N}$. Then we have canonical isomorphisms*

$$\begin{aligned}
&H_{\mathrm{de\ R}}(\Gamma^{\wedge}) \cong \mathbb{C}[\mathcal{D}^m, \mathcal{D}^{-m}] \otimes H_{\mathrm{de\ R}}(\Gamma^{\wedge}_{\mathrm{Inv}}) && \textit{for } \mathrm{GL}_q(N),\\
&H_{\mathrm{de\ R}}(\Gamma^{\wedge}) \cong H_{\mathrm{de\ R}}(\Gamma^{\wedge}_{\mathrm{Inv}}) \oplus \mathcal{D} H_{\mathrm{de\ R}}(\Gamma^{\wedge}_{\mathrm{Inv}}) && \textit{for } \mathrm{O}_q(2n+1),\ \Gamma = \Gamma_{\tau,1},\\
&H_{\mathrm{de\ R}}(\Gamma^{\wedge}) \cong H_{\mathrm{de\ R}}(\Gamma^{\wedge}_{\mathrm{Inv}}) \oplus H_{\mathrm{de\ R}}(\mathcal{D}\Gamma^{\wedge}_{\mathrm{Inv}}) && \textit{for } \mathrm{O}_q(2n+1),\\
&H_{\mathrm{de\ R}}(\Gamma^{\wedge}_{\mathrm{L}}) \cong H_{\mathrm{de\ R}}(\Gamma^{\wedge}_{\mathrm{R}}) \cong H_{\mathrm{de\ R}}(\Gamma^{\wedge}_{\mathrm{Inv}}) && \textit{in all cases.}
\end{aligned}$$

(c) *Let $G_q = \mathrm{O}_q(2n)$, $\mathcal{A} = \mathcal{O}(G_q)$, and Γ as in* (a)*. Then we have*

$$\begin{aligned}
H_{\mathrm{de\ R}}(\Gamma^{\wedge}) \cong{}& H_{\mathrm{de\ R}}(\Gamma^{\wedge}_{\mathrm{Inv}}) \oplus \mathcal{D} H_{\mathrm{de\ R}}(\Gamma^{\wedge}_{\mathrm{Inv}}) \oplus\\
&\oplus H_{\mathrm{de\ R}}(\Gamma^{\wedge}(\mathbf{1}, \mathcal{D})) \oplus H_{\mathrm{de\ R}}(\Gamma^{\wedge}(\mathcal{D}, \mathbf{1})),\\
H_{\mathrm{de\ R}}(\Gamma^{\wedge}_{\mathrm{L}}) \cong{}& H_{\mathrm{de\ R}}(\Gamma^{\wedge}_{\mathrm{Inv}}) \oplus H_{\mathrm{de\ R}}(\Gamma^{\wedge}(\mathbf{1}, \mathcal{D})),\\
H_{\mathrm{de\ R}}(\Gamma^{\wedge}_{\mathrm{R}}) \cong{}& H_{\mathrm{de\ R}}(\Gamma^{\wedge}_{\mathrm{Inv}}) \oplus H_{\mathrm{de\ R}}(\Gamma^{\wedge}(\mathcal{D}, \mathbf{1})).
\end{aligned}$$

REMARK 3.1. For $\mathrm{GL}_q(N)$ the case $z = 1$ is of special interest since the commutation relations between the differentials $\mathrm{d}u^i_j$ and the matrix elements u^m_n appear to be *linear*, i. e.

$$\mathrm{d}\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = \hat{R}^{\tau} \boldsymbol{u}_1 \cdot \mathrm{d}\boldsymbol{u}_2 \hat{R}^{\tau}. \tag{19}$$

These calculi were extensively studied in [**15**], [**11**] and [**16**]. In the proof of Lemma 6.5 given below we will show that $z = 1$ is regular.

THEOREM 3.2. *Suppose that q is a transcendental complex number. Let G_q be one of the quantum groups $\mathrm{GL}_q(N)$ or $\mathrm{SL}_q(N)$, and $\mathcal{A} =$*

$\mathcal{O}(G_q)$ its coordinate Hopf algebra. Let Γ be one of the N^2-dimensional bicovariant first order differential calculi $\Gamma_{\tau,z}$ over $\mathcal{A}$, where z is regular in case of $\mathrm{GL}_q(N)$. Then we have the Hodge decompositions

$$\begin{aligned}\Gamma^{\wedge k} &\cong \mathrm{d}\,\Gamma^{\wedge k-1} \oplus \partial^{+}\,\Gamma^{\wedge k+1} \oplus H^k_{\text{de R}}(\Gamma^{\wedge}),\\ \Gamma^{\wedge k} &\cong \mathrm{d}\,\Gamma^{\wedge k-1} \oplus \partial^{-}\,\Gamma^{\wedge k+1} \oplus H^k_{\text{de R}}(\Gamma^{\wedge})\end{aligned} \tag{20}$$

for $k \in \mathbb{N}_0$. Moreover, the cohomology ring of $\Gamma^{\wedge}$ is isomorphic to the algebra of coinvariant forms and to the vector space of harmonic forms:

$$H^k_{\text{de R}}(\Gamma^{\wedge}) \cong H^k_{\text{de R}}(\Gamma^{\wedge}_{\text{Inv}}) \cong \Gamma^{\wedge k}_{\text{Inv}} \cong H^{+}(\Gamma^{\wedge k}) \cong H^{-}(\Gamma^{\wedge k}). \tag{21}$$

Remarks 3.2. (i) In [**14**, Theorem 3.2] it was shown that $\Gamma^{\wedge}_{\text{Inv}}$ is a graded commutative algebra and its Poincaré series has the form $(1+t)(1+t^3)\cdots(1+t^{2N-1})$. By the above theorem, $\dim H^{N^2}_{\text{de R}}(\Gamma^{\wedge}) = \dim \Gamma^{\wedge N^2}_{\text{Inv}} = 1$. This means that there exists a linear functional f on $\mathcal{A}$ with the following property. For all $a \in \mathcal{A}$ there exists $\rho \in \Gamma^{\wedge(N^2-1)}$ such that

$$a\nu = \mathrm{d}\rho + f(a)\nu,$$

where $\nu \in \Gamma^{\wedge N^2}_{\text{Inv}}$ is the unique up to scalars coinvariant form of degree N^2 (volume form). From the fact that d is bicovariant, one derives easily that $a_{(1)}f(a_{(2)}) = f(a_{(1)})a_{(2)} = f(a)1$ and $f(1) = 1$. Therefore, f is the Haar functional h of the cosemisimple Hopf algebra $\mathcal{A}$.

(ii) If $\mathcal{A}$ belongs to the B-, C-, or D-series, then a coinvariant form is not closed in general. However, there is a weaker form of the decomposition (20). Let $\Lambda^k := L^{+}(\Gamma^{\wedge k}) = L^{-}(\Gamma^{\wedge k})$. Then one can prove that

$$\mathrm{d}\,\Lambda^{k-1} \oplus \partial^{+}\,\Lambda^{k+1} \cong \Lambda^k \cong \mathrm{d}\,\Lambda^{k-1} \oplus \partial^{-}\,\Lambda^{k+1}. \tag{22}$$

Using a computer algebra program we calculated the first terms of the Poincaré series $P(\Gamma^{\wedge}_{\text{Inv}}, t) = 1+t+5t^3+15t^4+\cdots$ and $P(H_{\text{de R}}(\Gamma^{\wedge}_{\text{Inv}}), t) = 1+t+2t^3+2t^4+\cdots$.

4. *Duality of Hopf bimodules*

We will show that the notion of a σ-metric naturally emerges by considering the left-dual and right-dual Hopf bimodules of a given Hopf bimodule. Our main result states that the left-dual Hopf bimodule is isomorphic to the right-dual Hopf bimodule. This makes the notion of a bicovariant σ-metric more transparent. However, both notions are not identical since the dual pairings g_{L} and g_{R} are not completely σ-symmetric while the metric g is.

Definition 4.1. Suppose that $\varGamma$ is a Hopf bimodule. A Hopf bimodule ${}^\vee\varGamma$ is called the *left-dual* to the Hopf bimodule $\varGamma$ if there exists a homomorphism $g_{\mathrm{L}}: {}^\vee\varGamma\otimes_{\mathcal{A}}\varGamma \to \mathcal{A}$ of Hopf bimodules such that the pairing g_{L} is non-degenerate. Similarly, a Hopf bimodule $\varGamma^\vee$ is called the *right-dual* to the Hopf bimodule $\varGamma$ if there exists a non-degenerate homomorphism $g_{\mathrm{R}}: \varGamma\otimes_{\mathcal{A}}\varGamma^\vee \to \mathcal{A}$ of Hopf bimodules.

Remarks 4.1. (i) Note that the left-dual ${}^\vee\varGamma$ and the right dual $\varGamma^\vee$ to the Hopf bimodule $\varGamma$ always exist. Moreover, they are unique up to isomorphisms. Indeed, the projection $P_{\mathrm{L}}\,\rho := S(\rho_{(1)})\rho_{(0)}$ onto the left-coinvariant subspace commutes with g_{L}. Hence ${}^\vee\varGamma_{\mathrm{L}}$ and $\varGamma_{\mathrm{L}}$ are dually paired vector spaces. Suppose that $\varGamma = (\boldsymbol{v}, \boldsymbol{f})$. It was shown in [**2**, Section II. 4] that ${}^\vee\varGamma = (\boldsymbol{v}^{\mathrm{c}}, {}^{\mathrm{c}}\boldsymbol{f})$. Similarly one proves that $\varGamma^\vee = ({}^{\mathrm{c}}\boldsymbol{v}, \boldsymbol{f}^{\mathrm{c}})$.

(ii) Since g_{L} and g_{R} are homomorphisms of bicomodules, they are bicovariant, i. e. $(\mathrm{id}\otimes g_{\mathrm{L}})\Delta_{\mathrm{L}} = \Delta g_{\mathrm{L}} = (g_{\mathrm{L}}\otimes \mathrm{id})\Delta_{\mathrm{R}}$ and $(\mathrm{id}\otimes g_{\mathrm{R}})\Delta_{\mathrm{L}} = \Delta g_{\mathrm{R}} = (g_{\mathrm{R}}\otimes \mathrm{id})\Delta_{\mathrm{R}}$ on ${}^\vee\varGamma\otimes_{\mathcal{A}}\varGamma$ and $\varGamma\otimes_{\mathcal{A}}\varGamma^\vee$, respectively.

(iii) It was shown in [**2**, Section II. 4] that the pairing is compatible with the braiding σ. More precisely, let $\varLambda$ be a Hopf bimodule. Then we have $g_{\mathrm{L}23}\sigma_{12}\sigma_{23} = g_{\mathrm{L}12}$ on ${}^\vee\varGamma\otimes_{\mathcal{A}}\varGamma\otimes_{\mathcal{A}}\varLambda$ and $g_{\mathrm{L}12}\sigma_{23}\sigma_{12} = g_{\mathrm{L}23}$ on $\varLambda\otimes_{\mathcal{A}}{}^\vee\varGamma\otimes_{\mathcal{A}}\varGamma$. Similarly, $g_{\mathrm{R}23}\sigma_{12}\sigma_{23} = g_{\mathrm{R}12}$ on $\varGamma\otimes_{\mathcal{A}}\varGamma^\vee\otimes_{\mathcal{A}}\varLambda$ and $g_{\mathrm{R}12}\sigma_{23}\sigma_{12} = g_{\mathrm{R}23}$ on $\varLambda\otimes_{\mathcal{A}}\varGamma\otimes_{\mathcal{A}}\varGamma^\vee$.

PROPOSITION 4.1. *Let $\mathcal{A}$ be a Hopf algebra with invertible antipode and let $\Gamma = (\boldsymbol{v}, \boldsymbol{f})$ be a Hopf bimodule with basis $\{\omega_i\}$ of $\Gamma_{\rm L}$. Let $\{\theta_i\}$ and $\{\eta_i\}$ denote the left-coinvariant bases of ${}^\vee\Gamma$ and $\Gamma^\vee$, dual to $\{\omega_i\}$, respectively.*
Then the linear mapping $T\colon {}^\vee\Gamma \to \Gamma^\vee$ defined by $T(a\,\theta_i) = a\,S^2(\eta_i) = a\,S(v^i_j)\eta_k\,v^j_k$ is an isomorphism of Hopf bimodules, where S denotes the antipode in the graded super Hopf algebra $(\Gamma^\vee)^\otimes$. Moreover, we have the following σ-symmetry of the above pairing:

$$g_{\rm R}(\mathrm{id}\otimes T) = g_{\rm L}\sigma \qquad \text{on} \qquad \Gamma\otimes_{\mathcal{A}}{}^\vee\Gamma.$$

Proof. (a) We first show that T is a right comodule mapping, i. e. $\Delta_{\rm R}(\tilde\theta_i) = \tilde\theta_j \otimes (\boldsymbol{v}^{\rm c})^j_i$ for $\tilde\theta_i := T(\theta_i) = T^j_i\,\eta_j$. Recall that the coproduct on $\Gamma^\vee$ is given by $\Delta(\eta_i) = \Delta_{\rm L}(\eta_i) + \Delta_{\rm R}(\eta_i) = 1\otimes\eta_i + \eta_j \otimes S^{-1}(v^i_j)$, see [**8**, Proposition 13.7]. Since $\varepsilon(\eta_i) = 0$ one has $S(\eta_i) = -\eta_j v^i_j$ and consequently,

$$\tilde\theta_i = S^2(\eta_i) = -Sv^i_j\,S(\eta_j) = Sv^i_j\,\eta_k\,v^j_k = \eta_k \triangleleft v^i_k. \qquad (23)$$

Hence $\Delta_{\rm R}\,\tilde\theta_i = Sv^x_j\,\eta_z\,v^j_y \otimes Sv^i_x\,S^{-1}v^k_z\,v^y_k = \tilde\theta_x \otimes Sv^i_x$ which proves (a).

(b) We show that T is a right module map. Since S^2 is an algebra map of $(\Gamma^\vee)^\otimes$ we have for $a\in\mathcal{A}$

$$\begin{aligned}\tilde\theta_i\,S^2(a) &= S^2(\eta_i\,a) = S^2\big((S(f^j_i)*a)\,\eta_j\big)\\ &= S^2\big(a_{(1)}Sf^j_i(a_{(2)})\big)\tilde\theta_j = (S^2a)_{(1)}S^{-1}f^j_i\big((S^2a)_{(2)}\big)\tilde\theta_j\\ &= (S^{-1}f^j_i * S^2a)\tilde\theta_j = T\big((S^{-1}f^j_i * S^2a)\,\theta_j\big)\\ &= T(\theta_i\,S^2(a)).\end{aligned}$$

Since $S^2\colon \mathcal{A}\to\mathcal{A}$ is surjective, T is a right module map. By (23), $\tilde\theta_i$ is a left-coinvariant 1-form. Hence T is a left comodule map. Therefore T is an isomorphism of Hopf bimodules. In particular, $\overline{T} := (T^l_i) \in \mathrm{Mor}(\boldsymbol{v}^{\rm c}, {}^{\rm c}\boldsymbol{v})$ and $\overline{T}^t \in \mathrm{Mor}(\boldsymbol{f}^{\rm c}, {}^{\rm c}\boldsymbol{f})$.

(c) We prove the last assertion. Since $\boldsymbol{f}^{\mathrm{c}}$ defines the right action on η_i, we obtain from (23) that $\tilde{\theta}_i = Sv^i_j(Sf^l_k * v^j_k)\eta_l = Sf^l_k(v^i_k)\eta_l$, i. e. $T^l_i = Sf^l_k(v^i_k)$. Since σ, g_{L}, g_{R}, and T are bimodule maps it suffices to prove the statement for $\omega_i \in \varGamma_{\mathrm{L}}$ and $\theta_j \in {}^{\vee}\varGamma_{\mathrm{L}}$. By (1)

$$\begin{aligned} g_{\mathrm{L}}\sigma(\omega_i \otimes_{\mathcal{A}} \theta_j) &= g_{\mathrm{L}}(\theta_k \otimes_{\mathcal{A}} (\omega_i \triangleleft Sv^j_k)) = g_{\mathrm{L}}(\theta_k \otimes_{\mathcal{A}} f^i_n(Sv^j_k)\,\omega_n) \\ &= f^i_n(Sv^j_n) = T^i_j = g_{\mathrm{R}}(\omega_i \otimes_{\mathcal{A}} T(\theta_j)). \end{aligned}$$

▮

Remarks 4.2. (i) Unfortunately the equation $g_{\mathrm{L}}\,(T^{-1} \otimes \mathrm{id}) = g_{\mathrm{R}}\,\sigma$ on $\varGamma^{\vee} \otimes_{\mathcal{A}} \varGamma$ is not fulfilled in general. If this symmetry holds, then the matrices $\overline{T}$ and $\tilde{T} = (\tilde{T}^a_b)$, $\tilde{T}^a_b = Sf^k_b(v^k_a)$, have to be inverse to each other. This is not the case for the fundamental Hopf bimodules $(\boldsymbol{u}, \boldsymbol{\ell}^{\pm\mathrm{c}})$, but for the differential Hopf bimodules $\varGamma_{\tau,z}$ it is. A sufficient condition for the second σ-symmetry is that $\varGamma = (\boldsymbol{v}, \boldsymbol{f})$ is an irreducible Hopf bimodule and that both $\varGamma_{\mathrm{Inv}}$ and $\varGamma^{\vee}_{\mathrm{Inv}}$ are nontrivial.

(ii) It is easy to show that ${}^{\vee}\varGamma_{+,z} \cong \varGamma_{-,z}$. Moreover, the σ-metric $g\colon \varGamma_{+,z} \otimes_{\mathcal{A}} \varGamma_{-,z} \to \mathcal{A}$, see (13), can be obtained from g_{R} by the above identification of $\varGamma_{-,z}$ with the right-dual Hopf bimodule $\varGamma^{\vee}_{+,z}$ of $\varGamma_{+,z}$:

$$g(\omega^+_{ij}, \omega^-_{kl}) = g(\omega^+_{ij}, T^{-1}(\omega^{\vee}_{kl})) := g_{\mathrm{R}}(\omega^+_{ij}, (T^{-1})^{mn}_{kl}\omega^{\vee}_{mn}) = (T^{-1})^{ji}_{kl},$$

where $\{\omega^{\vee}_{kl}\}$ is the left-coinvariant basis of $\varGamma^{\vee}_{+,z}$ such that $g_{\mathrm{R}}(\omega^+_{ij}, \omega^{\vee}_{mn}) = \delta_{jm}\delta_{in}$ and $T^{mn}_{kl} = (\ell^{-\mathrm{c}} \otimes \ell^+)^{mn}_{xy}\big(S(u^k_x\,(u^{\mathrm{c}})^l_y)\big)$ by step (c) of the above proof.

5. *Properties of the contraction*

We summarise some useful properties of the contraction, see [**6**, Lemmata 4.3, 4.4, 6.2]. For $\xi_i \in \varGamma^{\wedge k_i}_{\tau_i}$, $i = 0, 1, 2$, $\tau_1 = \tau_2 = -\tau_0$, $k_1 + k_2 \le k_0$, the contractions satisfy the following relations:

$$\begin{aligned} &\langle \xi_1, \langle \xi_2, \xi_0 \rangle_\pm \rangle_\pm = \langle \xi_1 \wedge \xi_2, \xi_0 \rangle_\pm,\ \langle \langle \xi_0, \xi_1 \rangle_\pm, \xi_2 \rangle_\pm = \langle \xi_0, \xi_1 \wedge \xi_2 \rangle_\pm, \\ &\langle \xi_1, \langle \xi_0, \xi_2 \rangle_\pm \rangle_\pm = \langle \langle \xi_1, \xi_0 \rangle_\pm, \xi_2 \rangle_\pm. \end{aligned} \tag{24}$$

For $a\in\mathcal{A}$, $\rho\in(\Gamma_\tau)_{\rm L}$, and $\zeta\in(\Gamma_{-\tau})_{\rm L}$ we have $h\langle a\rho,\zeta\rangle_\pm=h\langle\rho a,\zeta\rangle_\pm=h(a)\langle\rho,\zeta\rangle_\pm$. Since $\langle\cdot,\cdot\rangle_\pm$ is σ-symmetric, for $\xi\in\Gamma_\tau$ we particularly get

$$h\langle\xi,\omega_0^{-\tau}\rangle_\pm=h\langle\omega_0^{-\tau},\xi\rangle_\pm. \tag{25}$$

For $\xi\in\Gamma_\tau^{\wedge k}$, $\xi'\in\Gamma_{-\tau}^{\wedge k}$, $\rho_1\in\Gamma_\tau$, $\rho_2\in\Gamma_{-\tau}$, $k\geq 1$, we have

$$\begin{aligned}\langle\xi\wedge\rho_1,\rho_2\rangle_\pm&=\xi\langle\rho_1,\rho_2\rangle_\pm-\langle\xi,\rho_{(1)}^\mp\rangle_\pm\wedge\rho_{(2)}^\mp,\\ \langle\rho_1,\rho_2\wedge\xi'\rangle_\pm&=\langle\rho_1,\rho_2\rangle_\pm\xi'-\rho_{(1)}^\mp\wedge\langle\rho_{(2)}^\mp,\xi'\rangle_\pm,\end{aligned} \tag{26}$$

where $\sigma^\mp(\rho_1\otimes_{\mathcal{A}}\rho_2)=\rho_{(1)}^\mp\otimes_{\mathcal{A}}\rho_{(2)}^\mp\in\Gamma_{-\tau}\otimes_{\mathcal{A}}\Gamma_\tau$.

Now let us prove an identity for the braiding morphism σ.

LEMMA 5.1. *Let Γ and Λ be Hopf bimodules over $\mathcal{A}$. Then we have*

$$(\sigma_{\Gamma,\Lambda})_k\cdots(\sigma_{\Gamma,\Lambda})_1=\sigma_{\Gamma,\Lambda^{\otimes k}} \tag{27}$$

and this map is a homomorphism of the Hopf bimodule $\Gamma\otimes_{\mathcal{A}}\Lambda\otimes_{\mathcal{A}}\cdots\otimes_{\mathcal{A}}\Lambda$ to $\Lambda\otimes_{\mathcal{A}}\cdots\otimes_{\mathcal{A}}\Lambda\otimes_{\mathcal{A}}\Gamma$. Moreover, replacing $\Lambda^{\otimes k}$ by its quotient $\Lambda^{\wedge k}$, equation (27) remains valid. Similarly, $\sigma_k^-\cdots\sigma_1^-:\Gamma\otimes_{\mathcal{A}}\Lambda^{\wedge k}\to\Lambda^{\wedge k}\otimes_{\mathcal{A}}\Gamma$ is well-defined and coincides with $\sigma^-_{\Gamma,\Lambda^{\wedge k}}$.

Proof. (a) The braiding σ is compatible with the tensor product of Hopf bimodules in the sense that the identity $(\mathrm{id}_Y\otimes\sigma_{X,Z})\circ(\sigma_{X,Y}\otimes\mathrm{id}_Z)=\sigma_{X,Y\otimes Z}$ is fulfilled for all Hopf bimodules X, Y, and Z, see [**19**, Theorem 5.2]. Iterating this yields

$$(\sigma_{\Gamma,\Lambda})_k(\sigma_{\Gamma,\Lambda})_{k-1}\cdots(\sigma_{\Gamma,\Lambda})_1=\sigma_{\Gamma,\Lambda^{\otimes k}}.$$

(b) In what follows we skip the space indices Γ and Λ to simplify the notations. Since $\sigma_{i+1}\sigma_i\sigma_{i+1}=\sigma_i\sigma_{i+1}\sigma_i$ we obtain $\sigma_k\cdots\sigma_1\,\sigma_{i+1}=\sigma_i\,\sigma_k\cdots\sigma_1$, $i=1,\dots,k-1$. It follows that $\sigma_k\cdots\sigma_1\,(A_k)_{2\cdots k+1}=A_k\,\sigma_k\cdots\sigma_1$. Hence $\sigma_k\cdots\sigma_1$ maps $\Gamma\otimes_{\mathcal{A}}\ker A_k$ to $\ker A_k\otimes_{\mathcal{A}}\Gamma$ and therefore it defines a mapping $\Gamma\otimes_{\mathcal{A}}\Lambda^{\wedge k}\to\Lambda^{\wedge k}\otimes_{\mathcal{A}}\Gamma$. Moreover, by (27) it coincides with $\sigma_{\Gamma,\Lambda^{\wedge k}}$. Since σ^- defines a braiding as well and $\ker A_k=\ker A_k^-$, the proof for σ^- is analogous. ∎

Now we add some new relations which not yet appeared in [**6**].

Lemma 5.2. *For $\xi \in \Gamma_\tau^{\wedge k}$, $\rho_1 \in \Gamma_\tau$, $\rho_2 \in \Gamma_{-\tau}$, $k \geq 1$, $\tau \in \{+,-\}$, we have*

$$\begin{aligned} \langle \rho_1 \wedge \xi, \rho_2 \rangle_\pm &= \rho_1 \wedge \langle \xi, \rho_2 \rangle_\pm + (-1)^k \widetilde{g}(\sigma^\pm(\rho_1 \otimes_{\mathcal{A}} \xi), \rho_2), \\ \langle \rho_2, \xi \wedge \rho_1 \rangle_\pm &= \langle \rho_2, \xi \rangle_\pm \wedge \rho_1 + (-1)^k \widetilde{g}(\rho_2, \sigma^\pm(\xi \otimes_{\mathcal{A}} \rho_1)), \end{aligned} \tag{28}$$

where $\sigma^\pm = \sigma^\pm_{\Gamma_\tau, \Gamma_\tau^{\wedge k}}$ in the first equation and $\sigma^\pm = \sigma^\pm_{\Gamma_\tau^{\wedge k}, \Gamma_\tau}$ in the second equation.

Proof. We carry out the proof of the first equation. The proof of the second one is analogous. By definition (6),

$$\langle \rho_1 \wedge \xi, \rho_2 \rangle_\pm = \widetilde{g}\big((1 - \sigma^\pm_k + \sigma^\pm_k \sigma^\pm_{k-1} - \cdots + (-1)^k \sigma^\pm_k \cdots \sigma^\pm_1)(\rho_1 \otimes_{\mathcal{A}} \xi), \rho_2\big).$$

Note that the endomorphisms $\sigma^\pm_k$, $\sigma^\pm_k \sigma^\pm_{k-1}$, ... , $\sigma^\pm_k \cdots \sigma^\pm_2$ do *not* act in the first component. So we can separate the last summand. Applying Lemma 5.1 we continue

$$\begin{aligned} &= \widetilde{g}\big(\rho_1 \otimes_{\mathcal{A}} (1 - \sigma^\pm_{k-1} + \cdots + (-1)^{k-1} \sigma^\pm_{k-1} \cdots \sigma^\pm_1)\xi, \rho_2\big) + \\ &\quad + (-1)^k \widetilde{g}(\sigma^\pm_k \cdots \sigma^\pm_1 (\rho_1 \otimes_{\mathcal{A}} \xi), \rho_2) \\ &= \rho_1 \otimes_{\mathcal{A}} \langle \xi, \rho_2 \rangle_\pm + (-1)^k \widetilde{g}(\sigma^\pm(\rho_1 \otimes_{\mathcal{A}} \xi), \rho_2). \end{aligned}$$

This finishes the proof. ∎

6. *Quantum Laplace-Beltrami Operators*

Let us derive some important properties of the quantum Laplace-Beltrami operators defined by (15).

Proposition 6.1. *For $\rho \in \Gamma_\tau^{\wedge k}$ we have*

$$L^\pm_\tau \rho = (-1)^k \Big(-2\mathfrak{s}\rho + \widetilde{g}\big(\sigma^\pm(\omega_0^\tau \otimes_{\mathcal{A}} \rho), \omega_0^{-\tau}\big) + \widetilde{g}\big(\omega_0^{-\tau}, \sigma^\pm(\rho \otimes_{\mathcal{A}} \omega_0^\tau)\big)\Big). \tag{29}$$

Proof. Using the definition of d_τ and $\partial^\pm_\tau$, the identity $\sigma(\omega_0^{-\tau}\otimes_{\mathcal{A}}\omega_0^\tau) = \omega_0^\tau\otimes_{\mathcal{A}}\omega_0^{-\tau}$ and equations (26) and (14) we get (the not underlined terms remain unchanged)

$$\begin{aligned}
L^\pm_\tau\rho &= -\mathrm{d}_\tau\big(\langle\rho,\omega_0^{-\tau}\rangle_\pm + (-1)^k\langle\omega_0^{-\tau},\rho\rangle_\pm\big) + \partial^\pm_\tau\big(\omega_0^\tau\wedge\rho + (-1)^{k-1}\rho\wedge\omega_0^\tau\big)\\
&= -\omega_0^\tau\wedge\langle\rho,\omega_0^{-\tau}\rangle_\pm + (-1)^{k-1}\langle\rho,\omega_0^{-\tau}\rangle_\pm\wedge\omega_0^\tau + (-1)^{k+1}\big(\omega_0^\tau\wedge\langle\omega_0^{-\tau},\rho\rangle_\pm\\
&\quad + (-1)^k\langle\omega_0^{-\tau},\rho\rangle_\pm\wedge\omega_0^\tau\big) + \langle\omega_0^\tau\wedge\rho,\omega_0^{-\tau}\rangle_\pm + \underline{(-1)^{k+1}\langle\omega_0^{-\tau},\omega_0^\tau\wedge\rho\rangle_\pm}\\
&\quad + (-1)^{k-1}(\underline{\langle\rho\wedge\omega_0^\tau,\omega_0^{-\tau}\rangle_\pm} + (-1)^{k+1}\langle\omega_0^{-\tau},\rho\wedge\omega_0^\tau\rangle_\pm)\\
&= -\omega_0^\tau\wedge\langle\rho,\omega_0^{-\tau}\rangle_\pm + \underline{(-1)^{k-1}\langle\rho,\omega_0^{-\tau}\rangle_\pm\wedge\omega_0^\tau} + \underline{(-1)^{k+1}\omega_0^\tau\wedge\langle\omega_0^{-\tau},\rho\rangle_\pm}\\
&\quad - \langle\omega_0^{-\tau},\rho\rangle_\pm\wedge\omega_0^\tau + \langle\omega_0^\tau\wedge\rho,\omega_0^{-\tau}\rangle_\pm + (-1)^{k+1}(\mathfrak{s}\rho\underline{-\omega_0^\tau\wedge\langle\omega_0^{-\tau},\rho\rangle_\pm})\\
&\quad + (-1)^{k-1}(\mathfrak{s}\rho\underline{-\langle\rho,\omega_0^{-\tau}\rangle_\pm\wedge\omega_0^\tau}) + \langle\omega_0^{-\tau},\rho\wedge\omega_0^\tau\rangle_\pm\\
&= 2\mathfrak{s}(-1)^{k+1}\rho - \omega_0^\tau\wedge\langle\rho,\omega^{-\tau}\rangle_\pm - \langle\omega_0^{-\tau},\rho\rangle_\pm\wedge\omega_0^\tau + \underline{\langle\omega_0^\tau\wedge\rho,\omega_0^{-\tau}\rangle_\pm} +\\
&\quad + \underline{\langle\omega_0^{-\tau},\rho\wedge\omega_0^\tau\rangle_\pm}\\
&= 2\mathfrak{s}(-1)^{k+1}\rho + (-1)^k\big(\widetilde{g}(\sigma^\pm(\omega_0^\tau\otimes_{\mathcal{A}}\rho),\omega_0^{-\tau}) + \widetilde{g}(\omega_0^{-\tau},\sigma^\pm(\rho\otimes_{\mathcal{A}}\omega_0^\tau))\big).
\end{aligned}$$

The last equation follows by (28). ▮

Lemma 6.2. *For $a\in\mathcal{A}$ and $\tau\in\{+,-\}$ we have*

$$g(\omega^\tau(a),\omega_0^{-\tau}) = g(\omega_0^{-\tau},\omega^\tau(a)) = \mathfrak{s}X_0^\tau(a). \tag{30}$$

Proof. Let $\omega_{1,ij}^\tau := \omega_{ij}^\tau - (P_0^\tau)_{ij}^{kl}\omega_{kl}^\tau$, where P_0^τ is given by (12). Further let $X_{1,ij}^\tau$ be the corresponding dual basis elements in the quantum tangent space of Γ_τ. Since the σ-metric is bicovariant, the complex matrices $(g(\omega_{1,ij}^\tau,\omega_0^{-\tau}))$ and $(g(\omega_0^{-\tau},\omega_{1,ij}^\tau))$ are elements of the vector space $\mathrm{Mor}(\mathbf{1},\boldsymbol{v}\otimes\mathbf{1})$, where $\boldsymbol{v}$ denotes the corepresentation corresponding to the right coaction on $\langle\omega_{1,ij}^\tau\rangle$. By Schur's lemma these matrices have to be zero (see also the proof of Proposition 7.1 (i)). Using (14) and $\omega^\tau(a) = X_0^\tau(a)\omega_0^\tau + \sum_{ij}X_{1,ij}^\tau(a)\omega_{1,ij}^\tau$ the assertion follows. ▮

Proposition 6.3. *For $\rho \in \Gamma_\tau^{\wedge k}$, $\tau \in \{+,-\}$, we have*

$$\begin{aligned} L_\tau^+ \rho &= (-1)^k \mathfrak{s}\big(\rho_{(0)} X_0^\tau(\rho_{(1)}) + X_0^{-\tau}(\rho_{(-1)})\rho_{(0)}\big) \\ &= (-1)^k \mathfrak{s}\big(X_0^\tau * \rho + \rho * X_0^{-\tau}\big), \end{aligned} \tag{31}$$

$$\begin{aligned} L_\tau^- \rho &= (-1)^k \mathfrak{s}\big(\rho_{(0)} X_0^{-\tau}(\rho_{(1)}) + X_0^{\tau}(\rho_{(-1)})\rho_{(0)}\big) \\ &= (-1)^k \mathfrak{s}\big(X_0^{-\tau} * \rho + \rho * X_0^{\tau}\big). \end{aligned} \tag{32}$$

In particular, $L_\tau^+ a = L_\tau^- a = \mathfrak{s}X_0 * a = \mathfrak{s}a * X_0$ for $a \in \mathcal{A}$.

Proof. We prove (31). Let $\rho = \sum_i \rho_i a_i$ be a presentation of ρ with $\rho_i \in (\Gamma_\tau^{\wedge k})_{\mathrm{L}}$, $a_i \in \mathcal{A}$. By (1), since g and σ are $\mathcal{A}$-module homomorphisms and since $\beta a = a_{(1)}\beta \triangleleft a_{(2)}$, we obtain

$$\begin{aligned} \widetilde{g}(\sigma(\omega_0^\tau \otimes_{\mathcal{A}} \rho_i a_i), \omega_0^{-\tau}) &= \widetilde{g}(\sigma(\omega_0^\tau \otimes_{\mathcal{A}} \rho_i) a_i, \omega_0^{-\tau}) \\ &= \widetilde{g}(\rho_{i(0)} \otimes_{\mathcal{A}} (\omega_0^\tau \triangleleft \rho_{i(1)}) a_i, \omega_0^{-\tau}) \\ &= \widetilde{g}(\rho_{i(0)} a_{i(1)} \otimes_{\mathcal{A}} (\omega_0^\tau \triangleleft (\rho_{i(1)} a_{i(2)})), \omega_0^{-\tau}) \\ &= \rho_{i(0)} a_{i(1)} g\big(\omega^\tau(\rho_{i(1)} a_{i(2)}) + \varepsilon(\rho_{i(1)} a_{i(2)})\omega_0^\tau, \omega_0^{-\tau}\big) \\ &= \rho_{i(0)} a_{i(1)} X_0^\tau(\rho_{i(1)} a_{i(2)}) + \rho_i a_i g(\omega_0^\tau, \omega_0^{-\tau}) \\ &= \mathfrak{s}(\rho_{(0)} X_0^\tau(\rho_{(1)}) + \rho). \end{aligned}$$

In the fourth equation we used (9), in the fifth equation (30) and in the last one (14). Now let $\rho = \sum_i a_i \rho_i$, $\rho_i \in (\Gamma_\tau^{\wedge k})_{\mathrm{L}}$, $a_i \in \mathcal{A}$. By (30) and (14) we have

$$\begin{aligned} \widetilde{g}(\omega_0^{-\tau}, \sigma(a_i \rho_i \otimes_{\mathcal{A}} \omega_0^\tau)) &= \widetilde{g}(\omega_0^{-\tau} a_i, \sigma(\rho_i \otimes_{\mathcal{A}} \omega_0^\tau)) \\ &= \widetilde{g}(a_{i(1)} \omega_0^{-\tau} \triangleleft a_{i(2)}, \omega_0^\tau \otimes_{\mathcal{A}} \rho_i) \\ &= a_{i(1)} g\big(\omega^{-\tau}(a_{i(2)}) + \varepsilon(a_{i(2)})\omega_0^{-\tau}, \omega_0^\tau\big)\rho_i \\ &= \mathfrak{s}\big(a_{i(1)} X_0^{-\tau}(a_{i(2)})\rho_i + a_i \rho_i\big) \\ &= \mathfrak{s}(X_0^{-\tau}(\rho_{(-1)})\rho_{(0)} + \rho). \end{aligned}$$

In the last equation we used $b_{(1)} X_0^{-\tau}(b_{(2)}) = X_0^{-\tau}(b_{(1)}) b_{(2)}$ for all $b \in \mathcal{A}$ which follows from the centrality of $X_0^{-\tau}$. Inserting both parts into (29) we obtain (31).

Let us prove (32). Similarly to the preceding equation one shows that $\widetilde{g}(\sigma^-(\omega_0^\tau\otimes_{\mathcal{A}}\rho),\omega_0^{-\tau}) = \mathfrak{s}\big(\rho + X_0(\rho_{(-1)})\rho_{(0)}\big)$. Let $\rho = \sum_i a_i\rho_i$ with left-coinvariant elements ρ_i. Using (1) we get

$$\begin{aligned}
\widetilde{g}(\omega_0^{-\tau},&\sigma^-(a_i\rho_i\otimes_{\mathcal{A}}\omega_0^\tau)) = \widetilde{g}(\omega_0^{-\tau}a_i, (\omega_0^\tau \triangleleft S^{-1}\rho_{i(1)})\otimes_{\mathcal{A}}\rho_{i(0)})\\
&= g\big(\omega_0^{-\tau}a_i S(S^{-1}\rho_{i(2)}), \omega_0^\tau S^{-1}\rho_{i(1)}\big)\rho_{i(0)}\\
&= g\big(a_{i(1)}\rho_{i(2)}\omega^{-\tau}\triangleleft(a_{i(2)}\rho_{i(3)}), \omega_0^\tau\big)S^{-1}\rho_{i(1)}\rho_{i(0)}\\
&= \mathfrak{s}a_{i(1)}\rho_{i(2)}\big(\varepsilon(a_{i(2)}\rho_{i(3)}) + X_0^{-\tau}(a_{i(2)}\rho_{i(3)})\big)S^{-1}\rho_{i(1)}\rho_{i(0)}\\
&= \mathfrak{s}(\rho + a_{i(1)}\rho_{i(0)}X_0^{-\tau}(a_{i(2)}\rho_{i(1)}))\\
&= \mathfrak{s}(\rho + \rho_{(0)}X_0^{-\tau}(\rho_{(1)})).
\end{aligned}$$

This gives (32). Since X_0^τ is central, $L_\tau^+ a = L_\tau^- a = \mathfrak{s}X_0{*}a = \mathfrak{s}a{*}X_0$ follows from (31) and (32). ▮

Cosemisimple Hopf algebras. Let $\mathcal{A}$ be a cosemisimple Hopf algebra (cf. [**8**, Subsection 11.2]) and let $\widehat{\mathcal{A}}$ be the set of equivalence classes α of irreducible corepresentations $\boldsymbol{u}^\alpha$ of $\mathcal{A}$. Then $\mathcal{A}$ has the Peter-Weyl decomposition $\mathcal{A} = \bigoplus_{\alpha\in\widehat{\mathcal{A}}}\mathcal{C}(\boldsymbol{u}^\alpha)$. Let $P^\alpha\colon\mathcal{A}\to\mathcal{A}$ denote the projection of $\mathcal{A}$ onto the simple coalgebra $\mathcal{C}(\boldsymbol{u}^\alpha)$. In particular, if $a = \sum_{\lambda,i,j} c_{ij}^\lambda u_{ij}^\lambda$, $c_{ij}^\lambda\in\mathbb{C}$, and $\{u_{ij}^\lambda \,|\, i,\, j = 1,\dots,d_\lambda\}$ is a linear basis of $\mathcal{C}(\boldsymbol{u}^\lambda)$, then $P^\alpha(a) = \sum_{ij} c_{ij}^\alpha u_{ij}^\alpha$. Define the linear functionals h^α, $\alpha\in\widehat{\mathcal{A}}$, on $\mathcal{A}$ by $h^\alpha = \varepsilon\circ P^\alpha$. Obviously, we have $\sum_{\alpha\in\widehat{\mathcal{A}}} h^\alpha = \varepsilon$ and $(P^\alpha\otimes\mathrm{id})\Delta = (\mathrm{id}\otimes P^\alpha)\Delta = (P^\alpha\otimes P^\alpha)\Delta = \Delta\circ P^\alpha$. It is easily seen that $h^\alpha * a = a * h^\alpha = P^\alpha(a)$. Note that h^0 corresponding to $\boldsymbol{u}^0\equiv\mathbf{1}$ is the Haar functional on $\mathcal{A}$.

Since X_0^τ is central, for $\lambda\in\widehat{\mathcal{A}}$ there exist complex numbers E_λ^τ such that $X_0^\tau *$ acts as a scalar on $\mathcal{C}(\boldsymbol{u}^\lambda)$:

$$\begin{aligned}
\mathfrak{s}X_0^\tau * h^\lambda * a &= E_\lambda^\tau h^\lambda * a,\\
\mathfrak{s}X_0^\tau(h^\lambda * a) &= E_\lambda^\tau h^\lambda(a)
\end{aligned}\tag{33}$$

for $a \in \mathcal{A}$. Let $\boldsymbol{v}, \boldsymbol{w}$ be corepresentations of $\mathcal{A}$ and $\Gamma = \Gamma_\tau$. Define the following subspaces of $\Gamma^\wedge$:

$$\Gamma^k(\boldsymbol{v}, \boldsymbol{w}) := \{\rho \in \Gamma^{\wedge k} \mid \rho_{(-1)} \otimes \rho_{(0)} \otimes \rho_{(1)} \in \mathcal{C}(\boldsymbol{v}) \otimes \Gamma^{\wedge k} \otimes \mathcal{C}(\boldsymbol{w})\},$$
$$\Gamma^\wedge(\boldsymbol{v}, \boldsymbol{w}) := \bigoplus_{k \geq 0} \Gamma^k(\boldsymbol{v}, \boldsymbol{w}).$$

We briefly write $\Gamma^k(\lambda, \mu)$ instead of $\Gamma^k(\boldsymbol{u}^\lambda, \boldsymbol{u}^\mu)$. The main step in our proof is the following spectral decomposition of the quantum Laplace-Beltrami operators.

Proposition 6.4. *Let $\Gamma = \Gamma_\tau$, $\tau \in \{+,-\}$. For λ, $\mu \in \widehat{\mathcal{A}}$ define the mapping $h^{\lambda\mu} \colon \Gamma^\wedge \to \Gamma^\wedge(\lambda, \mu)$ by $h^{\lambda\mu}(\rho) = h^\lambda(\rho_{(-1)})\rho_{(0)}h^\mu(\rho_{(1)})$. For $\rho \in \Gamma^\wedge$ and $\rho^{\lambda\mu} = h^{\lambda\mu}(\rho)$ we then have*

$$\rho = \sum_{\lambda, \mu \in \widehat{\mathcal{A}}} \rho^{\lambda\mu}, \quad \Gamma^\wedge = \bigoplus_{k \geq 0} \bigoplus_{\lambda, \mu \in \widehat{\mathcal{A}}} \Gamma^k(\lambda, \mu), \tag{34}$$

$$\begin{aligned} L^+_\tau \rho^{\lambda\mu} &= (-1)^k (E^{-\tau}_\lambda + E^\tau_\mu) \rho^{\lambda\mu}, \\ L^-_\tau \rho^{\lambda\mu} &= (-1)^k (E^\tau_\lambda + E^{-\tau}_\mu) \rho^{\lambda\mu}. \end{aligned} \tag{35}$$

For brevity we write $E_{\lambda\mu} := E^-_\lambda + E^+_\mu$.

Proof. (a) An easy computation shows that indeed $\rho^{\lambda\mu} \in \Gamma^\wedge(\lambda, \mu)$. Since $\sum_\lambda h^\lambda = \varepsilon$ and $\rho = \varepsilon(\rho_{(-1)})\rho_{(0)}\varepsilon(\rho_{(1)})$, the first part of (34) follows. Let us verify the second part of (34). The first sum is direct by the grading. The second sum is direct, since matrix elements of inequivalent irreducible corepresentations are linearly independent.

(b) Since $\Delta_{\mathrm{R}}(\rho^{\lambda\mu}) = h^\lambda(\rho_{(-1)})\rho_{(0)} \otimes h^\mu * \rho_{(1)}$ and $\Delta_{\mathrm{L}}(\rho^{\lambda\mu}) = \rho_{(-1)} * h^\lambda \otimes \rho_{(0)} h^\mu(\rho_{(1)})$, by (31) and (33) we obtain the equation

$$\begin{aligned} L^+_\tau \rho^{\lambda\mu} =& (-1)^k \mathfrak{s}\big(h^\lambda(\rho_{(-1)})\rho_{(0)} X^\tau_0(h^\mu * \rho_{(1)}) + \\ &+ X^{-\tau}_0(h^\lambda * \rho_{(-1)})\rho_{(0)} h^\mu(\rho_{(1)})\big) = (-1)^k (E^\tau_\mu + E^{-\tau}_\lambda) \rho^{\lambda\mu}. \end{aligned}$$

The proof of the second part of (35) is analogous. ∎

In the remainder of this section $\mathcal{A}$ denotes the coordinate Hopf algebra of the quantum group G_q as in Theorem 3.1. For $\lambda \in P_{++}$ as usual λ'_i denotes the length of the i^{th}-column of λ. We define $P(\mathcal{A})$ to be the set P_+ for $\mathrm{GL}_q(N)$, $P^0_{++} := \{\lambda \in P_{++} \,|\, \lambda_N = 0\}$ for $\mathrm{SL}_q(N)$, $\{\lambda \in P_{++} \,|\, \lambda'_1 + \lambda'_2 \leq N\}$ for $\mathrm{O}_q(N)$, $\{\lambda \in P_{++} \,|\, \lambda'_1 \leq n\}$ for $\mathrm{Sp}_q(2n)$, and $\{\lambda \in P_{++} \,|\, \lambda'_1 \leq \frac{N}{2}\}$ for $\mathrm{SO}_q(N)$, respectively. By [**8**, Theorem 11.22] irreducible corepresentations $\boldsymbol{v}$ of $\mathcal{A}$ are in one-to-one correspondence with elements of $P(\mathcal{A})$. We identify $\widehat{\mathcal{A}}$ and $P(\mathcal{A})$.

Lemma 6.5. *Suppose that $\lambda, \mu \in P(\mathcal{A})$.*
(i) *For $\mathrm{SL}_q(N)$ we have $E_{\lambda\mu} = 0$ if and only if $\lambda = \mu = (0)$.*
(ii) *For $\mathrm{GL}_q(N)$ we have $E_{\lambda\mu} = F_{\lambda\mu}$. The parameter value $z = 1$ is regular. If $z^N q^{-2} = \zeta$ for a primitive m^{th} root of unity ζ, $m \in \mathbb{N}$, then we have $E_{\lambda\mu} = 0$ if and only if $\lambda = (n^N)$ and $\mu = (k^N)$ for some $n, k \in m\mathbb{Z}$.*
(iii) *In the cases $\mathrm{Sp}_q(N)$ and $\mathrm{SO}_q(N)$ we have $E_{\lambda\mu} = 0$ if and only if $\lambda = \mu = (0)$. In the case $\mathrm{O}_q(N)$ we have $E_{\lambda\mu} = 0$ if and only if $\lambda, \mu \in \{(0), (1^N)\}$.*

Proof. For $\tau \in \{+, -\}$ define the following rational functions of t and z:

$$
\begin{aligned}
e^\tau_\lambda(t,z) &:= z^{-\tau m}\big([\![N]\!]_t + \tau(t - t^{-1}) \sum_{x\in\lambda} \mathrm{sgn}(x) t^{\tau(N+2c(x))}\big) - [\![N]\!]_t, \\
e_{\lambda\mu}(t,z) &:= e^-_\lambda(t,z) + e^+_\mu(t,z), \qquad (36)
\end{aligned}
$$

where $m = |\lambda|$. It follows from [**6**, Proposition 7.1] that for the quantum groups $\mathrm{GL}_q(N)$ and $\mathrm{SL}_q(N)$ and for $\lambda \in P_{++}$ we have $E^\tau_\lambda = e^\tau_\lambda(q,z)$, where $z^N = q^2$ in the $\mathrm{SL}_q(N)$ case and $z \neq 0$ in the $\mathrm{GL}_q(N)$ case. Note that we have to replace z^2 in [**6**] by z according to our definition of $\Gamma_{\tau,z}$. Obviously, $E_{\lambda\mu} = e_{\lambda\mu}(q,z)$ for $\lambda, \mu \in P^0_{++}$. Later we will see that $E^\tau_\lambda = e^\tau_\lambda(q,z)$ for $\lambda \in P_+$ and not only for $\lambda \in P_{++}$.

We prove (i). Set $\widetilde{E}^\tau_\lambda = \lim_{t\to 1}(t-t^{-1})^{-2}\, e^\tau_\lambda(t,\, t^{2/N})$. In the remark to Proposition 7.1 in [**6**] it was noted that

$$\widetilde{E}_\lambda := \widetilde{E}^+_\lambda + \widetilde{E}^-_\lambda = \sum_{i=1}^{N-1} \frac{(N-i)m_i}{N}\big(i(m_i+N) + 2\sum_{j=1}^{i-1} jm_j\big), \tag{37}$$

where $m_i = \lambda_i - \lambda_{i+1}$, $i = 1,\dots,N-1$. On the other hand, computing the limit $\lim_{t\to 1}(t-t^{-1})^{-2} e^\tau_\lambda(t, t^{2/N})$ directly from (36) one gets

$$\widetilde{E}^+_\lambda = \widetilde{E}^-_\lambda = \frac{1}{2N}(mN^2 + 2c(\lambda)N - m^2). \tag{38}$$

Using the formulae $c(\lambda) = n(\lambda') - n(\lambda)$, $n(\lambda) = \sum_{i\ge 1}(i-1)\lambda_i$, and $n(\lambda') = \sum_{i\ge 1}\frac{1}{2}\lambda_i(\lambda_i - 1)$ from [**10**, Chapter 1], we obtain (37) from (38). From (37) it follows that $\widetilde{E}^+_\lambda \ge 0$ for $\lambda \in P^0_{++}$ and $\widetilde{E}^+_\lambda = 0$ if and only if $\lambda = (0)$. Suppose that $E_{\lambda\mu} = 0$ for some $\lambda, \mu \in P^0_{++}$. Since $e_{\lambda\mu}(t) := e_{\lambda\mu}(t, t^{2/N})$ is an algebraic function of t and $t = q$ is a transcendental root, $e_{\lambda\mu}(t) \equiv 0$. In particular

$$\lim_{t\to 1}(t-t^{-1})^{-2} e_{\lambda\mu}(t, t^{2/N}) = \widetilde{E}^-_\lambda + \widetilde{E}^+_\mu = 0. \tag{39}$$

Hence $\lambda = \mu = (0)$.

Let us prove (ii). We will show that $E^\tau_\lambda = e^\tau_\lambda(q, z)$ for arbitrary $\lambda \in P_+$. For this purpose we prove that $\mathcal{D}^n\, a$ is an eigenvector for $X^\tau_0 *$ if a is and we compute the corresponding eigenvalue. Suppose that $\mathfrak{s}X^\tau_0 * a = E^\tau a$ for a complex number E^τ. Since $\ell^{+i}_{\ j}(\mathcal{D}) = qx^{-N}\delta_{ij}$ and $\ell^{-i}_{\ j}(\mathcal{D}) = q^{-1}y^N\delta_{ij}$ we have

$$\omega^\tau(\mathcal{D}) = (q^{2\tau} z^{-\tau N} - 1)\omega^\tau_0. \tag{40}$$

Hence $\omega^\tau_0 \triangleleft \mathcal{D} = q^{2\tau} z^{-\tau N}\omega^\tau_0$. Acting from the right by $\mathcal{D}^{-1}$ gives $\omega^\tau_0 \triangleleft \mathcal{D}^{-1} = q^{-2\tau} z^{\tau N}\omega^\tau_0$. For $n \in \mathbb{Z}$ we thus have $\omega^\tau_0 \triangleleft \mathcal{D}^n = q^{2n\tau} z^{-nN\tau}\omega^\tau_0$. Since $\mathcal{D}^n$ is grouplike, $\omega^\tau(a) = \omega^\tau_0 \triangleleft a - \varepsilon(a)\omega^\tau_0$, and $\rho\, a = a_{(1)}\rho \triangleleft a_{(2)}$, we obtain

by (30) the following formulae for $n \in \mathbb{Z}$:

$$\begin{aligned}
\mathfrak{s}X_0^\tau * (\mathcal{D}^n a) &= \mathfrak{s}\mathcal{D}^n a_{(1)} X_0^\tau(\mathcal{D}^n a_{(2)}) \\
&= \mathcal{D}^n a_{(1)} g\big(\omega^\tau(\mathcal{D}^n a_{(2)}), \omega_0^{-\tau}\big) \\
&= \mathcal{D}^n a_{(1)} g(\omega_0^\tau \triangleleft (\mathcal{D}^n a_{(2)}), \omega_0^{-\tau}) - \mathcal{D}^n a_{(1)} \varepsilon(a_{(2)}) g(\omega_0^\tau, \omega_0^{-\tau}) \\
&= \mathcal{D}^n a_{(1)} g(q^{2n\tau} z^{-nN\tau} \omega_0^\tau \triangleleft a_{(2)}, \omega_0^{-\tau}) - \mathfrak{s}\mathcal{D}^n a \\
&= q^{2n\tau} z^{-nN\tau} \mathcal{D}^n a_{(1)} \big(g(\omega^\tau(a_{(2)}), \omega_0^{-\tau}) + \varepsilon(a_{(2)})\mathfrak{s}\big) - \mathfrak{s}\mathcal{D}^n a \\
&= \big(q^{2n\tau} z^{-nN\tau}(E^\tau + \mathfrak{s}) - \mathfrak{s}\big)\mathcal{D}^n a.
\end{aligned}$$

Since $\mathcal{D}$ corresponds to the weight (1^N) and $\mathfrak{s} = [\![N]\!]_q$ we have for $\lambda \in P_+$

$$E^\tau_{\lambda+(1^N)} + [\![N]\!]_q = q^{2\tau} z^{-N\tau}(E^\tau_\lambda + [\![N]\!]_q). \tag{41}$$

Next we will show that for $\lambda \in P_+$,

$$e^\tau_{\lambda+(1^N)}(t,z) + [\![N]\!]_t = t^{2\tau} z^{-N\tau}(e^\tau_\lambda(t,z) + [\![N]\!]_t). \tag{42}$$

For $\lambda \in P_+$ set $\hat{\lambda} = \{(i,j) \in \lambda + (1^N) \mid (i, j-1) \in \lambda\}$. How to obtain $\lambda + (1^N)$ from λ? In cases $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$ shift λ to the right by 1, then add one box $(i,1)$ for each $i = 1, \ldots, k$. In cases $0 \geq \lambda_{k+1} \geq \cdots \geq \lambda_N$ remove the box $(i,0)$ and then shift the remainder by 1 to the right. Since boxes with $\lambda_i < 0$ have negative sign and $|\lambda + (1^N)| = m + N$ we

obtain by (36)

$$
\begin{aligned}
e^\tau_{\lambda+(1^N)}(t,z) =& z^{-\tau(m+N)}\big(\llbracket N\rrbracket_t+\tau(t-t^{-1})\sum_{x\in\hat\lambda}\operatorname{sgn}(x)\,t^{\tau(N+2c(x))}+\\
&+\tau(t-t^{-1})\sum_{i=1}^N t^{\tau(N+2-2i)}\big)-\llbracket N\rrbracket_t\\
=& z^{-\tau m-\tau N}\big(\llbracket N\rrbracket_t+\tau(t-t^{-1})t^{2\tau}\sum_{x\in\lambda}\operatorname{sgn}(x)\,t^{\tau(N+2c(x))}+\\
&+\tau(t-t^{-1})(t^{\tau N}+t^{\tau(N-2)}+\dots+t^{\tau(-N+2)})\big)-\llbracket N\rrbracket_t\\
=& z^{-\tau m-\tau N}\big((1-t^{2\tau})\llbracket N\rrbracket_t+z^{\tau m}t^{2\tau}(e^\tau_\lambda(t,z)+\llbracket N\rrbracket_t)+\\
&+\tau(t-t^{-1})t^\tau\llbracket N\rrbracket_t\big)-\llbracket N\rrbracket_t\\
=& t^{2\tau}z^{-\tau N}(e^\tau_\lambda(t,z)+\llbracket N\rrbracket_t)-\llbracket N\rrbracket_t.
\end{aligned}
$$

In the last line we used $\tau t^\tau(t-t^{-1})=t^{2\tau}-1$. Since there exists $n\in\mathbb{N}$ such that $\lambda+(n^N)\in P_{++}$ and $E^\tau_{\lambda+(n^N)}=e^\tau_{\lambda+(n^N)}(q,z)$, from (41) and (42) we obtain $E^\tau_\lambda=e^\tau_\lambda(q,z)$. Comparing (36) and (17) yields $E_{\lambda\mu}=F_{\lambda\mu}$.

We will show that $z=1$ is regular. Suppose that $F_{\lambda\mu}=0$ for some λ, $\mu\in P_+$. Inserting $z=1$ into (17) we get $F_{\lambda\mu}=e_{\lambda\mu}(q,1)$. Since $e_{\lambda\mu}(t,1)$ is a rational function of t and q is a transcendental root of it, $e_{\lambda\mu}(t,1)\equiv 0$. In particular $0=\lim_{t\to1}(t-t^{-1})^{-1}e_{\lambda\mu}(t,1)=|\mu|-|\lambda|$. Set $m:=|\lambda|=|\mu|$. Further we have $0=\lim_{t\to1}(t-t^{-1})^{-2}e_{\lambda\mu}(t,1)$. Since

$$
\begin{aligned}
(t-t^{-1})^{-2}e_{\lambda\mu}(t,1)=&\sum_{x\in\mu}\operatorname{sgn}(x)t^{\frac N2+c(x)}\llbracket \tfrac N2+c(x)\rrbracket_t-\\
&-\sum_{x\in\lambda}\operatorname{sgn}(x)t^{-\frac N2-c(x)}\llbracket -\tfrac N2-c(x)\rrbracket_t
\end{aligned}\tag{$*$}
$$

the limit $t\to1$ gives $0=mN+c(\mu)+c(\lambda)$. Let us define $\widetilde{E}^+_\nu:=(2N)^{-1}(|\nu|N^2+2c(\nu)N-|\nu|^2)$ for $\nu\in P_+$. By (38), $\widetilde{E}^+_\nu\ge0$ for $\nu\in P^0_{++}$ and $\widetilde{E}^+_\nu=0$ if and only if $\nu=(0)$. It is easy to check that $\widetilde{E}^+_{\nu+(1^N)}=\widetilde{E}^+_\nu$ for $\nu\in P_+$. Hence $\widetilde{E}^+_\nu\ge0$, $\nu\in P_+$, and $\widetilde{E}^+_\nu=0$ if and only if $\nu=(n^N)$ for some $n\in\mathbb{Z}$. We conclude that $c(\lambda),c(\mu)\ge(2N)^{-1}(m^2-mN^2)$. Inserting this into equation $(*)$ we obtain $0\ge\frac1N m^2\ge0$, where equality

holds on the left hand side if and only if $\lambda = (n^N)$ and $\mu = (l^N)$ for some n, $l \in \mathbb{Z}$ and on the right hand side if and only if $m = 0$. Hence $\lambda = \mu = (0)$ and $z = 1$ is regular.

Finally consider the case when $z^N q^{-2} = \zeta$ and ζ is a primitive m^{th} root of unity ($m \in \mathbb{N}$). Let $\tilde{z} = (t^2\zeta)^{1/N}$. Observe that $e^{\tau}_{\lambda+(m^N)}(t, \tilde{z}) = e^{\tau}_{\lambda}(t, \tilde{z})$ by (42) and since $\zeta^m = 1$. Hence $E_{\lambda+(m^N),\mu+(m^N)} = F_{\lambda+(m^N),\mu+(m^N)} = F_{\lambda\mu}$ for $\lambda, \mu \in P_+$. Therefore the if part of the assertion holds. Moreover, for the only if part we can assume that $\lambda, \mu \in P_{++}$. Since q is a transcendental root of the algebraic function $f(t) := e_{\lambda\mu}(t, \tilde{z})$, we conclude that $f(t)$ has to be identically zero. Therefore $\lim_{t\to\pm1} f(t)$ has to be zero, and by (36) it follows that $|\mu|, |\lambda| \in mN\mathbb{Z}$. Then $\lim_{t\to 1} f(t)/(t^2-1) = 0$. Further, one can compute that $\lim_{t\to 1} t^{N-1} f(t)/(t^2-1)^2 = \widetilde{E}^{+}_{\mu} + \widetilde{E}^{+}_{\lambda}$ (see (38)). This sum is positive except for the case $\widetilde{E}^{+}_{\mu} = \widetilde{E}^{+}_{\lambda} = 0$. Moreover, the latter is equivalent to $\lambda = (k^N), \mu = (l^N)$ for some $k, l \in \mathbb{Z}$. Together with $|\mu|, |\lambda| \in mN\mathbb{Z}$ we get the assertion.

(iii) There exists an isomorphism of Hopf bimodules $\varGamma_{+,z}$ and $\varGamma_{-,z}$, see [**8**, Subsection 14.6.1]. In particular $X^{+}_{0} = X^{-}_{0}$ and consequently $E^{+}_{\lambda} = E^{-}_{\lambda} =: E_{\lambda}$. By (10) and the definition of $\boldsymbol{\ell}^{\pm}$ one obtains

$$
\begin{aligned}
&\mathfrak{s}X^{+}_{0}(u^{i_1}_{k_1}\cdots u^{i_m}_{k_m} P_{\lambda}{}^{k_1\cdots k_m}_{j_1\cdots j_m}) = \\
&\qquad = z^m(\hat{R}_m\hat{R}_{m-1}\cdots\hat{R}_2\hat{R}_1^2\hat{R}_2\cdots\hat{R}_m - I)^{i_1\cdots i_m\, i}_{k_1\cdots k_m\, k}(D^{-1})^k_i\, P_{\lambda}{}^{k_1\cdots k_m}_{j_1\cdots j_m},
\end{aligned}
$$

where $P_\lambda \in \mathrm{Mor}(\boldsymbol{u}^{\otimes m})$. Paying attention to the 2-admissible parameter $z \in \{-1, 1\}$, the choice of the coinvariant 1-form ω^{τ}_{0}, and the definition of the σ-metric it follows from the remark after Proposition 7.2 in [**6**] that $E_\lambda = e_\lambda(q)$, where

$$
e_\lambda(t) := \epsilon z^{|\lambda|}(t - t^{-1})^2 \sum_{x\in\lambda} [\![N - \epsilon + 2c(x)]\!]_t.
$$

Suppose that $E_{\lambda\mu} = 0$. Since $\varGamma^{\wedge}(\lambda, \mu) = \{0\}$ for $|\lambda| \not\equiv |\mu| \mod (2)$ we may assume $|\lambda| \equiv |\mu| \mod (2)$. Since $e_\lambda(t) + e_\mu(t)$ is a rational

function with transcendental root $t = q$, $e_\lambda + e_\mu \equiv 0$. In particular

$$\begin{aligned} 0 \overset{!}{=} & \lim_{t\to 1} \epsilon z^{|\lambda|}(t-t^{-1})^{-2}(e_\lambda(t)+e_\mu(t)) \\ =& (|\lambda|+|\mu|)(N-\epsilon)+2c(\lambda)+2c(\mu). \end{aligned} \tag{43}$$

Using the inequality $2c(\nu)N \geq |\nu|^2 - |\nu|N^2$, $\nu \in P_{++}$, from the proof of (ii) it is easily seen that for $\lambda \neq (0)$ in the case $\epsilon = -1$ and in the case ($\epsilon = 1$ and $|\lambda| > N$) we have $2c(\lambda) > -|\lambda|(N-\epsilon)$. Thus, by (43), $\lambda = \mu = (0)$ or ($\epsilon = 1$ and $|\lambda|, |\mu| \leq N$). In the latter case $2c(\lambda) \geq -|\lambda|^2 + |\lambda|$ where equality holds if and only if $\lambda = (1^k)$, $1 \leq k \leq N$. Inserting this into (43) yields $\lambda, \mu \in \{(0), (1^N)\}$. Indeed we have $E_{\lambda\mu} = 0$ in these cases. ▮

Proof of Theorem 3.1. Let $[\rho]$ denote the cohomology class of $\rho \in \Gamma^\wedge$. Since d is bicovariant and $h^\lambda * a = a * h^\lambda$, the differential d commutes with $h^{\lambda\mu}$, $\lambda, \mu \in \widehat{\mathcal{A}}$. In particular $h^{0,0}$ factorises to $h^{0,0*}\colon H_{\mathrm{de\ R}}(\Gamma^\wedge) \to H_{\mathrm{de\ R}}(\Gamma^\wedge_{\mathrm{Inv}})$. Since $h^{0,0}\iota = \mathrm{id}$, where ι is the embedding of $\Gamma^\wedge_{\mathrm{Inv}}$ into $\Gamma^\wedge$, $h^{0,0*}$ is surjective. We prove injectivity. Let $\rho \in H_{\mathrm{de\ R}}(\Gamma^\wedge)$, in particular $\mathrm{d}\rho = 0$, and suppose that $h^{0,0*}(\rho) = [\rho^{0,0}] = 0$.

(i) Consider first the case (a) of Theorem 3.1 with $\Gamma = \Gamma_{+,z}$. By Lemma 6.5, $E_{\lambda\mu} \neq 0$ if and only if $(\lambda, \mu) \neq (0, 0)$. From (15) and (35) we obtain

$$\begin{aligned} \rho^{\lambda\mu} &= (-1)^k E^{-1}_{\lambda\mu}(-\mathrm{d}\,\partial^+ + \partial^+\,\mathrm{d})\,\rho^{\lambda\mu}, \\ \rho^{\lambda\mu} &= (-1)^k E^{-1}_{\mu\lambda}(-\mathrm{d}\,\partial^- + \partial^-\,\mathrm{d})\,\rho^{\lambda\mu} \end{aligned} \tag{44}$$

for $(\lambda, \mu) \neq (0, 0)$. Since d commutes with $h^{\lambda\mu}$ we get $\mathrm{d}\rho^{\lambda\mu} = h^{\lambda\mu}(\mathrm{d}\rho) = h^{\lambda\mu}(0) = 0$. By (44), $[\rho^{\lambda\mu}] = (-1)^k E^{-1}_{\lambda\mu}[-\mathrm{d}\,\partial^+\rho^{\lambda\mu}] = 0$ for $(\lambda, \mu) \neq (0, 0)$ (coboundary). Hence $[\rho] = [\rho^{0,0}] + \sum_{(\lambda,\mu)\neq(0,0)} [\rho^{\lambda\mu}] = 0$ and $h^{0,0*}$ is injective. In the same way the restrictions of the map $h^{0,0}$ to the subspaces $\Gamma^\wedge_{\mathrm{L}}$ and $\Gamma^\wedge_{\mathrm{R}}$ yield isomorphisms $H_{\mathrm{de\ R}}(\Gamma^\wedge_{\mathrm{L}}) \cong H_{\mathrm{de\ R}}(\Gamma^\wedge_{\mathrm{Inv}})$ and $H_{\mathrm{de\ R}}(\Gamma^\wedge_{\mathrm{R}}) \cong H_{\mathrm{de\ R}}(\Gamma^\wedge_{\mathrm{Inv}})$, respectively. This proves (a) in case

$\varGamma = \varGamma_{+,z}$. For $\varGamma = \varGamma_{-,z}$ use $\rho^{\lambda\mu} = (-1)^k E^{-1}_{\lambda\mu}(-\mathrm{d}\,\partial^- + \partial^-\,\mathrm{d})\rho^{\lambda\mu}$ and $\rho^{\lambda\mu} = (-1)^k E^{-1}_{\mu\lambda}(-\mathrm{d}\,\partial^+ + \partial^+\,\mathrm{d})\rho^{\lambda\mu}$ to get the same result.

(ii) Observe that for $a \in \mathcal{A}$ with $\mathrm{d}a = 0$ we have

$$H_{\mathrm{de\ R}}(a\varLambda^\wedge) \cong aH_{\mathrm{de\ R}}(\varLambda^\wedge). \tag{45}$$

Consider the quantum group $\mathrm{GL}_q(N)$ and suppose that the parameter z satisfies the condition $z^N q^{-2} = \zeta$, where ζ is a primitive m^{th} root of unity, $m \in \mathbb{N}$. Note that $\mathrm{d}(\mathcal{D}^m) = 0$ by (40). Further we have $\varGamma_{\mathrm{L}} = \varGamma(0,0) \oplus \varGamma(0,(1,0,\dots,0,-1))$ by (7) and $\boldsymbol{u} \otimes \boldsymbol{u}^{\mathrm{c}} \cong \mathbf{1} \oplus \boldsymbol{u}^{(1,0,\dots,0,-1)}$. By the Littlewood-Richardson rule for tensor product representations of $\mathrm{GL}(N)$ a necessary condition for $\varGamma^\wedge(0,\mu) \neq \{0\}$ is $|\mu| = 0$. Since $\varGamma^\wedge = \mathcal{A}\,\varGamma^\wedge_{\mathrm{L}}$, $\varGamma^\wedge(\lambda,\mu) \neq \{0\}$ implies $|\lambda| = |\mu|$. Combining this with Lemma 6.5, $E_{\lambda\mu} = 0$ is an eigenvalue of $L^\pm_\tau$ if and only if $\lambda = \mu = (n^N)$ for some $n \in m\mathbb{Z}$. Similarly as in (i) it follows that

$$h_{\mathcal{D}} := \sum_{n \in m\mathbb{Z}} h^{(n^N),(n^N)} \colon \varGamma^\wedge \to \bigoplus_{n \in m\mathbb{Z}} \varGamma^\wedge((n^N),(n^N)) = \bigoplus_{n \in m\mathbb{Z}} \mathcal{D}^n \varGamma^\wedge_{\mathrm{Inv}}$$

defines an isomorphism $h^*_{\mathcal{D}} \colon H_{\mathrm{de\ R}}(\varGamma^\wedge) \to \bigoplus_{n \in m\mathbb{Z}} H_{\mathrm{de\ R}}(\mathcal{D}^n \varGamma^\wedge_{\mathrm{Inv}})$. By (45), $H_{\mathrm{de\ R}}(\mathcal{D}^n \varGamma^\wedge_{\mathrm{Inv}}) = \mathcal{D}^n H_{\mathrm{de\ R}}(\varGamma^\wedge_{\mathrm{Inv}})$ for $n \in m\mathbb{Z}$. Since the images of both mappings $h_{\mathcal{D}} \restriction \varGamma^\wedge_{\mathrm{L}}$ and $h_{\mathcal{D}} \restriction \varGamma^\wedge_{\mathrm{R}}$ belong to $\varGamma^\wedge_{\mathrm{Inv}}$, they define quasi-isomorphisms from $\varGamma^\wedge_{\mathrm{L}}$ to $\varGamma^\wedge_{\mathrm{Inv}}$ and from $\varGamma^\wedge_{\mathrm{R}}$ to $\varGamma^\wedge_{\mathrm{Inv}}$, respectively. This proves the $\mathrm{GL}_q(N)$ part of (b).

(iii) Consider now the $\mathrm{O}_q(N)$ case. Since $\varGamma_{+,z} \cong \varGamma_{-,z}$, it suffices to carry out the proof for the calculus $\varGamma = \varGamma_{+,z}$, $z \in \{-1,1\}$. By Lemma 6.5, $E^{\lambda\mu} = 0$ is an eigenvalue of $L^+ = Lapm[]$ if and only if $\lambda,\mu \in \{(0),(1^N)\}$. Similarly as in (i) $h_{\mathcal{D}} := h^{0,0} + h^{(0),(1^N)} + h^{(1^N),(0)} + h^{(1^N),(1^N)}$ defines a quasi-isomorphism of $\varGamma^\wedge$ onto $\varGamma^\wedge(\mathbf{1},\mathbf{1}) + \varGamma^\wedge(\mathbf{1},\mathcal{D}) + \varGamma^\wedge(\mathcal{D},\mathbf{1}) + \varGamma^\wedge(\mathcal{D},\mathcal{D})$. By the definition of $\boldsymbol{\ell}^\pm$, $\hat{R}$, and $\mathcal{D}$ we obtain $\ell^{+i}_{\ j}(\mathcal{D}) = x^{-N}\delta_{ij}$ and $\ell^{-i}_{\ j}(\mathcal{D}) = y^N\delta_{ij}$. Consequently, $X^+_{ij}(\mathcal{D}) = (z^{-N}-1)\delta_{ij}$ and $\mathrm{d}\mathcal{D} = (z^{-N}-1)\mathcal{D}\omega^+_0$. Hence $\mathrm{d}\mathcal{D} \neq 0$ in case of $\mathrm{O}_q(2n+1)$ and $\varGamma_{\tau,-1}$. Otherwise $\mathrm{d}\mathcal{D} = 0$. Since $\boldsymbol{u} \otimes \boldsymbol{u}^{\mathrm{c}} \cong \boldsymbol{u}^{(2)} \oplus \boldsymbol{u}^{(11)} \oplus \mathbf{1}$, each irreducible subcorepresentation of any tensor

power $(\boldsymbol{u}\otimes\boldsymbol{u}^{\mathrm{c}})^{\otimes k}$ corresponds to a Young diagram with an *even* number of boxes. Consequently, $\Gamma^\wedge(\lambda,\mu)=\{0\}$ if $|\lambda|+|\mu|$ is odd. In particular, $\Gamma^\wedge(\mathbf{1},\mathcal{D})=\Gamma^\wedge(\mathcal{D},\mathbf{1})=\{0\}$ if N is odd. For even N however these spaces may be nonzero. This completes the proof of Theorem 3.1.

7. *Proof of Theorem 3.2*

First we will show that the duality of d_τ and $\partial^\pm_{-\tau}$ holds in a rather general setup. Secondly, we will prove that for the quantum groups $\mathrm{SL}_q(N)$ and $\mathrm{GL}_q(N)$ the differential calculi $\Gamma^\wedge_+$ and $\Gamma^\wedge_-$ are weakly isomorphic. Combining both we obtain the proof of the second theorem.

Duality of differential and codifferential.

Proposition 7.1. *Suppose that $\mathcal{A}$ is a cosemisimple Hopf algebra, (Γ_+,Γ_-) is a dual pair of bicovariant differential calculi and λ, $\mu\in\widehat{\mathcal{A}}$. For $\nu\in\widehat{\mathcal{A}}$ let $\nu^{\mathrm{c}}\in\widehat{\mathcal{A}}$ denote the class of the contragredient corepresentation $(\boldsymbol{u}^\nu)^{\mathrm{c}}$.*
(i) *The map $h\circ\langle\cdot,\cdot\rangle_\pm\colon\Gamma^k_\tau(\lambda,\mu)\times\Gamma^k_{-\tau}(\lambda^{\mathrm{c}},\mu^{\mathrm{c}})\to\mathbb{C}$, $\tau\in\{+,-\}$, $k\geq 0$, is non-degenerate.*
(ii) *The restricted differential $\mathrm{d}_\tau\colon\Gamma^k_\tau(\lambda,\mu)\to\Gamma^{k+1}_\tau(\lambda,\mu)$ is the dual operator to the restricted codifferential $\partial^\pm_{-\tau}\colon\Gamma^{k+1}_{-\tau}(\lambda^{\mathrm{c}},\mu^{\mathrm{c}})\to\Gamma^k_{-\tau}(\lambda^{\mathrm{c}},\mu^{\mathrm{c}})$ with respect to the pairing $h\circ\langle\cdot,\cdot\rangle_\pm$.*

Proof. (i) Set $\Gamma^\wedge:=\Gamma^\wedge_\tau$ and fix $\lambda\in\widehat{\mathcal{A}}$, $k\in\mathbb{N}$. Let us prove that $\sum_{\mu\in\widehat{\mathcal{A}}}\Gamma^k(\lambda,\mu)$ is finite dimensional. The space $\Gamma_{\mathrm{L}}^{\wedge k}\equiv\sum_{\nu\in\widehat{\mathcal{A}}}\Gamma^k(0,\nu)$ is finite dimensional since Γ_{L} is. Suppose that $\rho\in\sum_{\mu\in\widehat{\mathcal{A}}}\Gamma^k(\lambda,\mu)$. Since $\rho=\rho_{(-2)}\cdot S(\rho_{(-1)})\rho_{(0)}$, we deduce that $\rho\in\mathcal{C}(\boldsymbol{u}^\lambda)\Gamma_{\mathrm{L}}^{\wedge k}$. Because $\mathcal{C}(\boldsymbol{u}^\lambda)$ is finite dimensional, the assertion follows. Similarly, $\dim\sum_{\lambda\in\widehat{\mathcal{A}}}\Gamma^k(\lambda,\mu)<\infty$. For ν, $\kappa\in\widehat{\mathcal{A}}$ let $\{\rho^i_j\}$ and $\{\zeta^m_n\}$ denote

the linear bases of $\Gamma^k_\tau(\lambda,\mu)$ and $\Gamma^k_{-\tau}(\nu,\kappa)$, respectively. Then we have

$$(\mathrm{id}\otimes\Delta_{\mathrm{R}})\Delta_{\mathrm{L}}(\rho^i_j) = u^\lambda_{ix}\otimes\rho^x_y\otimes u^\mu_{yj},$$
$$(\mathrm{id}\otimes\Delta_{\mathrm{R}})\Delta_{\mathrm{L}}(\zeta^k_l) = u^\nu_{ka}\otimes\zeta^a_b\otimes u^\kappa_{bl}.$$

Set $h^{ik}_{jl} := h\langle\rho^i_j,\zeta^k_l\rangle_\pm$. By the left covariance of $\langle\cdot,\cdot\rangle_\pm$ it follows that

$$u^\lambda_{ix}u^\nu_{ky}h\langle\rho^x_j,\zeta^y_l\rangle_\pm = (\mathrm{id}\otimes h)\Delta\langle\rho^i_j,\zeta^k_l\rangle_\pm = 1{\cdot}h\langle\rho^i_j,\zeta^k_l\rangle_\pm,$$
$$u^\lambda_{ix}u^\nu_{ky}h^{xy}_{jl} = h^{ik}_{jl}1.$$

Hence $(h^{ik}_{jl})_{i,k}\in\mathrm{Mor}(\mathbf{1},\boldsymbol{u}^\lambda\otimes\boldsymbol{u}^\nu)$ for all j,l. By Schur's lemma we obtain $(h^{ik}_{jl})_{i,k}=0$ for $\boldsymbol{u}^\nu\not\cong\boldsymbol{u}^{\lambda^{\mathrm{c}}}$. Using right covariance, in a similar way we get $(h^{ik}_{jl})_{j,l}\in\mathrm{Mor}(\boldsymbol{u}^\mu\otimes\boldsymbol{u}^\kappa,\mathbf{1})$ for all i,k. Again by Schur's lemma $(h^{ik}_{jl})_{j,l}=0$ for $\boldsymbol{u}^\kappa\not\cong\boldsymbol{u}^{\mu^{\mathrm{c}}}$. Suppose now that $h\langle\rho,\zeta\rangle_\pm=0$ for a fixed $\rho\in\Gamma^k_\tau(\lambda,\mu)$ and all $\zeta\in\Gamma^k_{-\tau}(\lambda^{\mathrm{c}},\mu^{\mathrm{c}})$. By the above arguments $h\langle\rho,\zeta\rangle_\pm=0$ for all $\zeta\in\Gamma^k_{-\tau}(\nu,\kappa)$, $\nu,\kappa\in\widehat{\mathcal{A}}$, i.e. for all $\zeta\in\Gamma^{\wedge k}_{-\tau}$. Since the Haar functional is regular, i.e. $h(ab)=0$ for all $a\in\mathcal{A}$ implies $b=0$ and $h(ab)=0$ for all $b\in\mathcal{A}$ implies $a=0$, and since the pairing $\langle\cdot,\cdot\rangle_\pm\colon\Gamma^{\wedge k}_\tau\otimes_{\mathcal{A}}\Gamma^{\wedge k}_\tau\to\mathcal{A}$ is non-degenerate, the pairing $h\circ\langle\cdot,\cdot\rangle_\pm\colon\Gamma^{\wedge k}_\tau\otimes_{\mathcal{A}}\Gamma^{\wedge k}_{-\tau}\to\mathbb{C}$ is also non-degenerate, cf. [**6**, Section 6]. Therefore $\rho=0$. Non-degeneracy in the second component can be proved similarly.

(ii) Suppose that $\rho\in\Gamma^k_\tau(\lambda,\mu)$ and $\zeta\in\Gamma^{k+1}_{-\tau}(\lambda^{\mathrm{c}},\mu^{\mathrm{c}})$. Because of (24), (25), the σ-symmetry of g, and since $\langle\rho,\zeta\rangle_\pm\in\Gamma_{-\tau}$, we obtain

$$\begin{aligned}
h\langle\mathrm{d}\rho,\zeta\rangle_\pm &= h\langle\omega^\tau_0\wedge\rho-(-1)^k\rho\wedge\omega^\tau_0,\zeta\rangle_\pm\\
&= h\langle\omega^\tau_0,\langle\rho,\zeta\rangle_\pm\rangle_\pm+(-1)^{k+1}h\langle\rho,\langle\omega^\tau_0,\zeta\rangle_\pm\rangle_\pm\\
&= h\langle\langle\rho,\zeta\rangle_\pm,\omega^\tau_0\rangle_\pm+(-1)^{k+1}h\langle\rho,\langle\omega^\tau_0,\zeta\rangle_\pm\rangle_\pm\\
&= h\langle\rho,\langle\zeta,\omega^\tau_0\rangle_\pm\rangle_\pm+(-1)^{k+1}h\langle\rho,\langle\omega^\tau_0,\zeta\rangle_\pm\rangle_\pm\\
&= h\langle\rho,\partial^\pm\zeta\rangle_\pm.
\end{aligned}$$

The proof is complete. ▮

Homomorphy of differential calculi. In this subsection we define and study the notion of homomorphic differential calculi. Our aim is to show that for the quantum groups $\mathrm{GL}_q(N)$ and $\mathrm{SL}_q(N)$ the differential calculi $\Gamma_+^\wedge$ and $\Gamma_-^\wedge$ are weakly isomorphic in the following sense. There exists a Hopf algebra automorphism F of $\mathcal{A}$ which can be extended to a graded algebra isomorphism $F\colon \Gamma_+^\wedge \to \Gamma_-^\wedge$ such that $F\mathrm{d}_+ = \mathrm{d}_- F$.

Suppose that $f\colon \mathcal{B} \to \mathcal{C}$ is an algebra homomorphism and Λ is a $\mathcal{C}$-bimodule. Then Λ is a $\mathcal{B}$-bimodule via $a \cdot \eta \cdot b := f(a)\eta f(b)$.

Definition 7.1. (i) Let $\mathcal{B}$ and $\mathcal{C}$ be algebras and let (Γ, d_1) and (Λ, d_2) be first order differential calculi over $\mathcal{B}$ and $\mathcal{C}$, respectively. The pair (f, F) of an algebra homomorphism $f\colon \mathcal{B} \to \mathcal{C}$ and a $\mathcal{B}$-bimodule homomorphism $F\colon \Gamma \to \Lambda$ is called a *homomorphism of the first order differential calculi Γ and Λ* if

$$F\mathrm{d}_1 = \mathrm{d}_2 f. \tag{46}$$

(ii) Let $\Gamma^\wedge$ and $\Lambda^\wedge$ be differential calculi over $\mathcal{B}$ and $\mathcal{C}$, respectively. A graded algebra homomorphism $F\colon \Gamma^\wedge \to \Lambda^\wedge$ is called a *homomorphism of the differential calculi* $\Gamma^\wedge$ and $\Lambda^\wedge$ if

$$F\mathrm{d}_1 = \mathrm{d}_2 F. \tag{47}$$

Recall that two differential calculi $\Gamma^\wedge$ and $\Lambda^\wedge$ over the same algebra $\mathcal{B}$ are isomorphic *in the strong sense* if and only if there exists a bijective homomorphism $F\colon \Gamma^\wedge \to \Lambda^\wedge$ of differential calculi with $F_0 = \mathrm{id}$.

The next lemma characterises homomorphic differential calculi in terms of their associated right ideals and in terms of their quantum tangent spaces.

Lemma 7.2. *Let Γ and Λ be left-covariant first order differential calculi over the Hopf algebras $\mathcal{A}$ and $\mathcal{B}$ with associated right ideals $\mathcal{R}$ and $\mathcal{S}$ and quantum tangent spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. Suppose that $f\colon \mathcal{A} \to \mathcal{B}$ is a Hopf algebra homomorphism. The following are equivalent:*

(i) $f(\mathcal{R}) \subseteq \mathcal{S}$.

(ii) $f^t(\mathcal{Y}) \subseteq \mathcal{X}$.

(iii) *There exists a unique homomorphism* (f, F) *of the left-covariant FODC* Γ *and* Λ.

Proof. (iii)$\to$(i). Fix $r \in \mathcal{R}$. Since f is a Hopf algebra homomorphism and $F(a\mathrm{d}b) = f(a)\mathrm{d}f(b)$, we have

$$\begin{aligned} \omega_\Lambda(f(r)) &= S\big(f(r)_{(1)}\big)\mathrm{d}f(r)_{(2)} = S(f(r_{(1)}))\mathrm{d}f(r_{(2)}) \\ &= f(Sr_{(1)})F(\mathrm{d}r_{(2)}) = F(\omega(r)) = 0. \end{aligned} \tag{48}$$

Moreover, $\varepsilon(f(r)) = \varepsilon(r) = 0$. Hence $f(r) \in \mathcal{S}$.

(i)$\to$(ii). Since $\mathcal{S}$ and $\mathcal{Y}$ are orthogonal subspaces with respect to the pairing of $\mathcal{A}$ and $\mathcal{A}^\circ$, we have $f^t(Y)(r) = Y(f(r)) \in Y(\mathcal{S}) = \{0\}$ for all $r \in \mathcal{R}$ and $Y \in \mathcal{Y}$. Furthermore, $f^t(Y)(1) = Y(f(1)) = Y(1) = 0$. Hence $f^t(Y) \in \mathcal{X}$.

(ii)$\to$(iii). F is uniquely determined by (46), since $F(a_i\mathrm{d}b_i) := f(a_i)\mathrm{d}f(b_i)$. We show that F is well-defined. Let $\{\omega_i\}$ and $\{\eta_j\}$ be linear bases of Γ_{L} and Λ_{L}, respectively, and let $\{X_i\}$ and $\{Y_j\}$ be the corresponding dual bases of $\mathcal{X}$ and $\mathcal{Y}$, respectively. By assumption there exist $\alpha_{ij} \in \mathbb{C}$ such that $f^t(Y_j) = \alpha_{ij}X_i$. Suppose that $a_i\mathrm{d}b_i = 0$. Then we have $0 = a_i(X_k * b_i)\omega_k$ and consequently $0 = a_i(X_k * b_i)$ for all k. Using this fact, we conclude that

$$\begin{aligned} f(a_i)\mathrm{d}f(b_i) &= f(a_i)f(b_{i(1)})\omega_\Lambda(f(b_{i(2)})) = f(a_i)f(b_{i(1)})Y_j(f(b_{i(2)}))\eta_j \\ &= f(a_ib_{i(1)})f^t(Y_j)(b_{i(2)})\eta_j = \alpha_{kj}f(a_ib_{i(1)})X_k(b_{i(2)})\eta_j \\ &= \alpha_{kj}f(a_i(X_k * b_i))\eta_j = 0. \end{aligned}$$

Hence F is well-defined. ∎

The next lemma is straightforward to prove using covariance of d and the properties of F_0. We omit the proof.

Lemma 7.3. *Suppose that $\Gamma^\wedge$ and $\Lambda^\wedge$ are left-covariant differential calculi over the Hopf algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $F\colon \Gamma^\wedge \to \Lambda^\wedge$ be a homomorphism of differential calculi and F_0 a Hopf algebra homomorphism. Then we have $(F_0\otimes F)\Delta_{\mathrm{L}} = \Delta_{\mathrm{L}}F$ and $F(\rho\triangleleft a) = F\rho\triangleleft F_0 a$ for $\rho\in\Gamma^\wedge$ and $a\in\mathcal{A}$.*

Replacing left-covariance by right-covariance in the above lemma the first assertion reads as $(F\otimes F_0)\Delta_{\mathrm{R}} = \Delta_{\mathrm{R}}F$. Now we shall apply the new notion to our main example.

Proposition 7.4. *Let G_q be one of the quantum groups $\mathrm{GL}_q(N)$ or $\mathrm{SL}_q(N)$ and $\mathcal{A}=\mathcal{O}(G_q)$. For $k=1,\dots,N$, set $k'=N+1-k$.*

(i) *There exists a unique bijective homomorphism $F\colon \Gamma_+^\wedge \to \Gamma_-^\wedge$ of differential calculi such that*

$$F(u^a_b) = Su^{b'}_{a'},\quad a,b=1,\dots,N. \tag{49}$$

(ii) *For all $\lambda,\mu\in\widehat{\mathcal{A}}$ the restriction of F to $\Gamma_+^k(\lambda,\mu)$ is a bijection onto $\Gamma_-^k(\lambda^{\mathrm{c}},\mu^{\mathrm{c}})$.*

Proof. (a) First it is to show that there exists a Hopf algebra automorphism $F\colon\mathcal{A}\to\mathcal{A}$ which satisfies (49). To do this we prove that F preserves the relations of the Hopf algebra $\mathcal{A}$. It is easily shown that $\hat{R}^{ab}_{rs} = \hat{R}^{s'r'}_{b'a'}$ and $d_k^{-1} = d_{k'}$. Moreover, the q-antisymmetric tensor satisfies $\varepsilon_{i_1\cdots i_N} = \varepsilon_{i'_N\cdots i'_1}$. We show that the algebra homomorphism $f\colon \mathbb{C}\langle u^i_j\rangle \to \mathbb{C}\langle v^i_j\rangle$, given by $f(u^a_b) = v^{b'}_{a'}$ maps the generating relations appearing in the definition of the Hopf algebra $\mathcal{A}$ to those of the Hopf algebra $\mathcal{A}^{\mathrm{op,cop}}$. Here we assume that $\mathbb{C}\langle v^i_j\rangle$ has both opposite multiplication and opposite comultiplication. By the above identity for the matrix $\hat{R}$ we have

$$f(\hat{R}^{ab}_{xy}u^x_r\otimes u^y_s - u^a_x\otimes u^b_y\hat{R}^{xy}_{rs}) = \hat{R}^{y'x'}_{b'a'}v^{r'}_{x'}\otimes v^{s'}_{y'} - v^{x'}_{a'}\otimes v^{y'}_{b'}\hat{R}^{s'r'}_{y'x'}.$$

The right hand side generates the relations of the bialgebra $\mathcal{A}^{\mathrm{op,cop}}$. Similarly one shows consistency with the q-determinant relation. Fi-

nally we have $(f\otimes f)\Delta u^a_b = v^{x'}_{a'}\otimes v^{b'}_{x'} = \Delta v^{b'}_{a'} = \Delta(f(u^a_b))$ and $\varepsilon(f(u^a_b)) = \delta_{ab} = \varepsilon(u^a_b)$. Hence f is a homomorphism of bialgebras. Since both $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op,cop}}$ are Hopf algebras, f is a Hopf algebra homomorphism. Since the antipode is a Hopf algebra map of $\mathcal{A}^{\mathrm{op,cop}} \to \mathcal{A}$, $F = S\circ f$ is a Hopf algebra automorphism. Its inverse F^{-1} is given by $F^{-1}(u^a_b) = S^{-1}(u^{b'}_{a'})$.

Next we show that $F(\mathcal{C}(\boldsymbol{u}^\lambda)) = \mathcal{C}(\boldsymbol{u}^{\lambda^c})$. Let $P^\lambda \in \mathrm{Mor}(\boldsymbol{u}^{\otimes k})$ be a primitive idempotent such that $\mathcal{C}(\boldsymbol{u}^\lambda) = \langle (P^\lambda)^{\vec{n}}_{\vec{x}}\, u^{\vec{x}}_{\vec{m}} \mid \vec{n}, \vec{m} \in \{1,\dots,N\}^k\rangle$. For $\vec{x} = (x_1,\dots,x_k)$ we write $\vec{x}\,' = (x'_1,\dots,x'_k)$ and $\overleftarrow{x}' = (x'_k,\dots,x'_1)$. Let us show that $(Q^\lambda)^{\vec{n}}_{\vec{m}} := (P^\lambda)^{\overleftarrow{m}'}_{\overleftarrow{n}'}$ is a projector equivalent to P^λ. Idempotents P and Q are called equivalent, if there exist A, $B \in \mathrm{Mor}(\boldsymbol{u}^{\otimes k})$ such that $AB = P$ and $BA = Q$. For this let α and β denote the algebra automorphism and algebra antiautomorphism of $\mathrm{Mor}(\boldsymbol{u}^{\otimes k})$ defined by

$$\alpha(\hat{R}_{n,n+1}) = \hat{R}_{k-n,k-n+1} \quad \text{and} \quad \beta(\hat{R}_{n,n+1}) = \hat{R}_{n,n+1},$$

$n = 1,\dots,k-1$, respectively. By the theory of Hecke algebras it is easy to see that α and β map each twosided ideal of $\mathrm{Mor}(\boldsymbol{u}^{\otimes k})$ into itself. In particular, the image of a primitive idempotent is an equivalent primitive idempotent. By induction on k we will show that

$$T^{\overleftarrow{m}'}_{\overleftarrow{n}'} = \beta(\alpha(T))^{\vec{n}}_{\vec{m}}. \tag{50}$$

It is well known that for $T \in \mathrm{Mor}(\boldsymbol{u}^{\otimes k})$ there exist X, $Y \in \mathrm{Mor}(\boldsymbol{u}^{\otimes 2})$ and $B \in \mathrm{Mor}(\boldsymbol{u}^{\otimes k-1})$ such that $T = X_{k-1,k}BY_{k-1,k}$. Since X and Y can be written in terms of $\hat{R}$, $\hat{R}^{-1}$, and I, and since $\hat{R}^{ab}_{rs} = \hat{R}^{s'r'}_{b'a'}$, we have by induction assumption

$$\begin{aligned} T^{m'_k\cdots m'_1}_{n'_k\cdots n'_1} &= X^{m'_2m'_1}_{x'y'}B^{m'_k\cdots m'_3x'}_{n'_k\cdots n'_3z'}Y^{z'y'}_{n'_2n'_1} = Y^{n_1n_2}_{yz}\beta(\alpha(B))^{zn_3\cdots n_k}_{xm_3\cdots m_k}X^{yx}_{m_1m_2} \\ &= (Y_{12}\beta(\alpha(B))X_{12})^{\vec{n}}_{\vec{m}} = \beta(\alpha(X_{k-1,k}BY_{k-1,k}))^{\vec{n}}_{\vec{m}}. \end{aligned}$$

The character of $\boldsymbol{u}^\lambda$ is $\chi(\boldsymbol{u}^\lambda) = (P^\lambda)^{\vec{n}}_{\vec{x}}\, u^{\vec{x}}_{\vec{n}}$ [**14**, Lemma 5.1]. Since $Q^\lambda = \beta(\alpha(P^\lambda))$ is an equivalent idempotent, the corresponding characters of

the corepresentations $(P^\lambda)^{\vec{n}}_{\vec{x}}\, u^{\vec{x}}_{\vec{m}}$ and $(Q^\lambda)^{\vec{n}}_{\vec{x}}\, u^{\vec{x}}_{\vec{m}}$ coincide [**14**, Lemma 5.1]. We conclude that

$$\chi(F\boldsymbol{u}^\lambda) = (P^\lambda)^{\vec{n}}_{\vec{x}}\,(u^{\mathrm{c}})^{\vec{x}\,\prime}_{\vec{n}'} = (Q^\lambda)^{\overleftarrow{x}\,\prime}_{\overleftarrow{n}\,\prime}\, S(u^{\overleftarrow{n}\,\prime}_{\overleftarrow{x}\,\prime}) = S(\chi(\boldsymbol{u}^\lambda)) = \chi(\boldsymbol{u}^{\lambda^{\mathrm{c}}}). \quad (51)$$

Since F is a Hopf algebra homomorphism, matrix elements of irreducible corepresentations are mapped into each other. Hence $F(\mathcal{C}(\boldsymbol{u}^\lambda)) = \mathcal{C}(\boldsymbol{u}^{\lambda^{\mathrm{c}}})$.

(b) We show that $F^t(\mathcal{X}^-) \subseteq \mathcal{X}^+$. Since F is a coalgebra homomorphism, F^t is multiplicative on the subalgebra of $\mathcal{A}^\circ$ generated by the matrix elements $\ell^{\pm a}_{\ \ b}$, $a, b = 1, \dots, N$. We compute $F^t(\ell^{\pm a}_{\ \ b})$ on the generators of $\mathcal{A}$.

$$\begin{aligned} F^t(\ell^{\pm a}_{\ \ b})(u^r_s) &= \ell^{\pm a}_{\ \ b}(Fu^r_s) = \ell^{\pm a}_{\ \ b}(Su^{s'}_{r'}) \\ &= p^{\pm 1}(\hat{R}^{\mp 1})^{s'a}_{br'} = p^{\pm 1}(\hat{R}^{\mp 1})^{rb'}_{a's} = \ell^{\pm b'}_{\ \ a'}(Su^r_s) = (S\ell^{\pm b'}_{\ \ a'})(u^r_s), \end{aligned}$$

where $p = x$ in the $\boldsymbol{\ell}^+$-case and $p = y$ in the $\boldsymbol{\ell}^-$-case. Since both $F^t(\ell^{\pm a}_{\ \ b})$ and $S\ell^{\pm b'}_{\ \ a'}$ are representations of $\mathcal{A}$, we obtain $F^t(\ell^{\pm a}_{\ \ b}) = S\ell^{\pm b'}_{\ \ a'}$. By [**4**, Theorem 9.1] we have

$$\ell^{+r}_{\ \ v}\, S\ell^{-w}_{\ \ s}\hat{R}^{vb}_{wc} = \hat{R}^{rv}_{sw} S\ell^{-b}_{\ \ v}\, \ell^{+w}_{\ \ c}.$$

Multiplying this relation by D^s_r and noting that $D^s_r\hat{R}^{rv}_{sw} = \mathfrak{r}\delta_{vw}$, we get

$$D^s_r\ell^{+r}_{\ \ v}S\ell^{-w}_{\ \ s}\hat{R}^{vb}_{wc} = \mathfrak{r}S\ell^{-b}_{\ \ v}\ell^{+v}_{\ \ c}.$$

Multiplying the latter by $(\hat{R}^{-1})^{mc}_{na}(D^{-1})^a_b D^n_k$ and using the identity $\hat{R}^{vb}_{wc}(\hat{R}^{-1})^{mc}_{na}(D^{-1})^a_b D^n_k = \delta_{vn}\delta_{wm}$ we obtain

$$D^s_r\ell^{+r}_{\ \ k}S\ell^{-m}_{\ \ s} = \mathfrak{r}S\ell^{-b}_{\ \ v}\ell^{+v}_{\ \ c}(\hat{R}^{-1})^{mc}_{na}(D^{-1})^a_b D^n_k. \quad (52)$$

By (10), (52), $(\hat{R}^{-1})^{jc}_{na}(D^{-1})^a_c = \mathfrak{r}^{-1}\delta_{jn}$, and $d_k^{-1} = d_{k'}$, we then have

$$\begin{aligned}
F^t(X^-_{i'j'}) &= F^t\big((D^{-1})^k_l S^{-1}(\ell^{+}{}^{i'}_{k})\ell^{-}{}^{l}_{j'}\big) - (D^{-1})^{i'}_{j'} \\
&= (D^{-1})^k_l S^{-1}(S\ell^{+}{}^{k'}_{i})S\ell^{-}{}^{j}_{l'} - D^i_j \\
&= D^{l'}_{k'}\ell^{+}{}^{k'}_{i} S\ell^{-}{}^{j}_{l'} - D^i_j \\
&= \mathfrak{r}S\ell^{-}{}^{b}_{v}\ell^{+}{}^{v}_{c}(\hat{R}^{-1})^{jc}_{na}(D^{-1})^a_b D^n_i - D^i_j \\
&= \mathfrak{r}(X^+_{bc} + \delta_{bc})(\hat{R}^{-1})^{jc}_{na}(D^{-1})^a_b D^n_i - D^i_j \\
&= \mathfrak{r}X^+_{bc}(\hat{R}^{-1})^{jc}_{na}(D^{-1})^a_b D^n_i.
\end{aligned}$$

This completes the proof of (b). Note that $F^t\colon \mathcal{X}^- \to \mathcal{X}^+$ is bijective since $X^+_{ab} = \mathfrak{r}^{-1}F^t(X^-_{i'j'})(D^{-1})^i_k \hat{R}^{ka}_{lb} D^l_j$. By Lemma 7.2, $F(a\mathrm{d}b) := F(a)\mathrm{d}F(b)$ is a well-defined $\mathcal{A}$-module map from Γ_+ to Γ_-. Similarly, $F^{-1}\colon \Gamma_- \to \Gamma_+$, $F^{-1}(a\mathrm{d}b) := F^{-1}(a)\mathrm{d}F^{-1}(b)$ is a well-defined $\mathcal{A}$-module map inverse to F.

(c) Consider now higher order forms. Let $\tau \in \{+,-\}$ and F^τ denote F for $\tau = +$ and F^{-1} for $\tau = -$. Since F^τ is an $\mathcal{A}$-bimodule map, we can extend F^τ to an algebra map $F^\tau\colon \Gamma^{\otimes}_\tau \to \Gamma^{\otimes}_{-\tau}$. We prove that $F\sigma = \sigma F$ in $\Gamma_+^{\otimes 2}$. Since both F and σ are $\mathcal{A}$-bimodule maps and $\Gamma_+\otimes_{\mathcal{A}}\Gamma_+$ is a free left $\mathcal{A}$-module with basis $(\Gamma_+\otimes_{\mathcal{A}}\Gamma_+)_{\mathrm{L}}$, it suffices to prove this equation for left-coinvariant elements. Let ρ, $\xi \in (\Gamma_+)_{\mathrm{L}}$. By (1) and Lemma 7.3 we have

$$\begin{aligned}
F\sigma(\rho\otimes_{\mathcal{A}}\xi) &= F(\xi_{(0)}\otimes_{\mathcal{A}}(\rho\triangleleft\xi_{(1)})) = F(\xi_{(0)})\otimes_{\mathcal{A}}F(\rho\triangleleft\xi_{(1)}) \\
&= (F\xi)_{(0)}\otimes_{\mathcal{A}}(F\rho\triangleleft(F\xi)_{(1)}) = \sigma(F\rho\otimes_{\mathcal{A}}F\xi).
\end{aligned}$$

Hence F commutes with the antisymmetriser A_k, $k \in \mathbb{N}$. The same is true for F^{-1}. Consequently, $F^\tau\colon \Gamma^\wedge_\tau \to \Gamma^\wedge_{-\tau}$ is a well-defined algebra map and $F\,F^{-1} = F^{-1}F = \mathrm{id}$. Now let us prove assertion (ii). Let $\rho \in \Gamma^k_+(\lambda,\mu)$. By Lemma 7.3 and the last part of (a), $F\rho \in \Gamma^k_-(\lambda^{\mathrm{c}},\mu^{\mathrm{c}})$. Since $F^{-1}\colon \Gamma^\wedge_- \to \Gamma^\wedge_+$ also satisfies the assumptions of Lemma 7.3, it follows immediately that $F^{-1} \restriction \Gamma^k_-(\lambda^{\mathrm{c}},\mu^{\mathrm{c}})$ is inverse to $F \restriction \Gamma^k_+(\lambda,\mu)$. ▮

Remark 7.1. For the B-, C-, and D-series the differential calculi $\Gamma_+^\wedge$ and $\Gamma_-^\wedge$ are isomorphic (in the strong sense: $F_0 = \mathrm{id}$).

Now we are able to finish the proof of Theorem 3.2. By [**14**, Theorem 3.2 (iii)] the differential d vanishes on $\Gamma_{\mathrm{Inv}}^{\wedge k}$. Hence $H^k_{\mathrm{de\ R}}(\Gamma_{\mathrm{Inv}}^\wedge) \cong \Gamma_{\mathrm{Inv}}^{\wedge k}$. Combining this with Theorem 3.1 gives (21). Since $\Gamma_{\mathrm{Inv}}^{\wedge k} = \Gamma^k(0,0)$ it follows from (34) that $H^k_{\mathrm{de\ R}}(\Gamma^\wedge)$ is a direct summand in (20). By (44), for $(\lambda,\mu) \neq (0,0)$ we have the following formulae:

$$\begin{aligned} \Gamma_\tau^k(\lambda,\mu) &= \mathrm{d}\Gamma_\tau^{k-1}(\lambda,\mu) + \partial^+\Gamma_\tau^{k+1}(\lambda,\mu), \\ \Gamma_\tau^k(\lambda,\mu) &= \mathrm{d}\Gamma_\tau^{k-1}(\lambda,\mu) + \partial^-\Gamma_\tau^{k+1}(\lambda,\mu). \end{aligned} \tag{53}$$

We have to prove that both sums are direct. Since all vector spaces appearing in (53) are finite dimensional, it suffices to compare their dimensions. We denote the restriction of a linear map $f\colon \Gamma^\wedge \to \Gamma^\wedge$ to the space $\Gamma^k(\lambda,\mu)$ by $f^{k,\lambda,\mu}$. By Proposition 7.1 we have $\operatorname{rank} \partial_\tau^{\pm,k+1,\lambda,\mu} = \operatorname{rank} \mathrm{d}_{-\tau}^{k,\lambda^{\mathrm{c}},\mu^{\mathrm{c}}}$. Indeed, both $\mathrm{d}_{-\tau}^{k,\lambda^{\mathrm{c}},\mu^{\mathrm{c}}}$ and $\partial_\tau^{\pm,k+1,\lambda,\mu}$ are linear mappings acting on finite dimensional vector spaces and they are dual to each other. By Proposition 7.4, $\mathrm{d}_-^{k,\lambda^{\mathrm{c}},\mu^{\mathrm{c}}} F = F\mathrm{d}_+^{k,\lambda,\mu}$ and F is bijective. We conclude that $\operatorname{rank} \partial_\tau^{\pm,k+1,\lambda,\mu} = \operatorname{rank} \mathrm{d}_{-\tau}^{k,\lambda^{\mathrm{c}},\mu^{\mathrm{c}}} = \operatorname{rank} \mathrm{d}_\tau^{k,\lambda,\mu}$. Since $H^k_{\mathrm{de\ R}}(\Gamma_\tau^\wedge(\lambda,\mu)) = \{0\}$, $\dim\ker \mathrm{d}_\tau^{k,\lambda,\mu} = \operatorname{rank} \mathrm{d}_\tau^{k-1,\lambda,\mu}$. Finally we obtain

$$\begin{aligned} \dim \Gamma_\tau^k(\lambda,\mu) &= \dim\ker \mathrm{d}_\tau^{k,\lambda,\mu} + \operatorname{rank} \mathrm{d}_\tau^{k,\lambda,\mu} \\ &= \operatorname{rank} \mathrm{d}_\tau^{k-1,\lambda,\mu} + \operatorname{rank} \partial_\tau^{\pm,\,k+1,\lambda,\mu}. \end{aligned}$$

It follows that the sums (53) are direct. The proof of Theorem 3.2 is complete.

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